42nd Annual Virgina Tech Regional Math Contest

From 9:00am – 11:30am

October 22, 2022

1. Give all possible representations of 2022 as a sum of at least two consecutive positive integers and prove that these are the only representations.

Let $S_{M,N} = \sum_{n=M}^{N} n$ be the sum of consecutive positive integers from $n = M$ to $n = N$ and recall the formula

$$S_{1,N} = \frac{N(N+1)}{2}$$

We aim to solve the equation $S_{M,N} = 2022$. Using the above formula, observe

$$\begin{aligned} S_{M,N} &= S_{1,N} - S_{1,M-1} \\ &= \frac{N(N+1)}{2} - \frac{(M-1)M}{2} \\ &= \frac{N^2 + (M+N) - M^2}{2} \end{aligned}$$

Hence, an equivalent formulation for our equation is

$$\underbrace{N^2 + (M+N) - M^2}_{(N^2-M^2)+(N+M)} = 4044 \quad \Longrightarrow \quad (N+M)(N-M+1) = 2^2 \cdot 3 \cdot 337$$

$N + M$ and $N - M + 1$ must have opposite parity. This leads to checking 6 possible cases, of which we can quickly eliminate the possibilities corresponding to $N + M \in \{3, 4, 12\}$:

(a) **Case 1:** $N + M = 337$, $N - M + 1 = 12$
The system of equations reduces to $2N = 348$, or $N = 174$. Then we find $M = 163$. Thus, $S_{163,174} = 2022$.

(b) **Case 2:** $N + M = 1011$, $N - M + 1 = 4$
The system of equations reduces to $2N = 1014$, or $N = 507$. Then we find $M = 504$. Thus, $S_{504,507} = 2022$.

(c) **Case 3:** $N + M = 1348$, $N - M + 1 = 3$
The system of equations reduces to $2N = 1350$, or $N = 675$. Then we find $M = 673$. Thus, $S_{673,675} = 2022$.

2. Let A and B be the two foci of an ellipse and let P be a point on this ellipse. Prove that the focal radii of P (that is, the segments $\overline{AP}$ and $\overline{BP}$) form equal angles with the tangent to the ellipse at P.

Let ℓ be the line passing through P such that AP and BP form equal angles with ℓ. It suffices to show that the line ℓ is tangent to the ellipse.

If ℓ is not tangent to the ellipse, then it intersects the ellipse both at P and at a point Q different from P. Let A' be the reflection of A with respect to the line ℓ. Since AP and BP form equal angles with ℓ, the points A', P, and B are collinear. It follows that

$$|A'B| = |A'P| + |PB| = |AP| + |PB| = |AQ| + |QB| = |A'Q| + |QB|.$$

By the triangle inequality, this is impossible unless A', Q, and B are on the same line, i.e. $P = Q$, a contradiction.

3. Find all positive integers a, b, c, d, and n satisfying $n^a + n^b + n^c = n^d$ and prove that these are the only such solutions.

Without loss of generality, $a \leq b \leq c \leq d$. Then dividing through by n^d, we have

$$n^{a-d} + n^{b-d} + n^{c-d} = 1$$

Each of the exponents is at most -1, so we have

$$n^{a-d} + n^{b-d} + n^{c-d} \leq \frac{3}{n}$$

which forces $1 \leq n \leq 3$. Clearly $n \neq 1$.

For $n = 3$, we must have all exponents equal to -1; that is, $a - d = -1$, $b - d = -1$, and $c - d = -1$. Hence, for any natural number $k \geq 2$, it follows that

$$\begin{aligned} a &= k - 1 \\ b &= k - 1 \\ c &= k - 1 \\ d &= k \end{aligned}$$

gives a solution to the equation.

For $n = 2$, we can have neither $\min\{a-d, b-d, c-d\} \geq -1$ nor $\max\{a-d, b-d, c-d\} \leq -2$, hence $c - d = -1$. The equation then reduces to

$$n^{a-d} + n^{b-d} = \frac{1}{2}$$

and by a similar argument, we must have $a - d = -2$ and $b - d = -2$. Then, for any natural number $k \geq 3$, it follows that

$$\begin{aligned} a &= k - 2 \\ b &= k - 2 \\ c &= k - 1 \\ d &= k \end{aligned}$$

4. Calculate the exact value of the series $\sum_{n=2}^{\infty} \log(n^3+1) - \log(n^3-1)$ and provide justification.

Using properties of logarithms, we have

$$\begin{aligned}\sum_{n=2}^{\infty} \log(n^3+1) - \log(n^3-1) &= \sum_{n=2}^{\infty} \log\left(\frac{n^3+1}{n^3-1}\right) \\ &= \log\left(\prod_{n=2}^{\infty} \frac{n^3+1}{n^3-1}\right) \\ &= \log\left(\prod_{n=2}^{\infty} \frac{(n+1)(n^2-n+1)}{(n-1)(n^2+n+1)}\right) \\ &= \log\left(\lim_{N\to\infty} \prod_{n=2}^{N} \frac{(n+1)(n^2-n+1)}{(n-1)(n^2+n+1)}\right)\end{aligned}$$

Then the product satisfies

$$\prod_{n=2}^{N} \frac{n+1}{n-1} = \frac{(3)(4)(5)(6)(7)\cdots(N+1)}{(1)(2)(3)(4)(5)\cdots(N-1)} = \frac{1}{2}\cdot\left(\frac{N+1}{N-1}\right)$$

and

$$\prod_{n=2}^{N} \frac{n^2-n+1}{n^2+n+1} = \frac{(3)(7)(13)(21)(31)\cdots(N^2-N+1)}{(7)(13)(21)(31)\cdots(N^2+N+1)} = 3\cdot\left(\frac{N^2-N+1}{N^2+N+1}\right)$$

Letting $N \to \infty$, we have

$$\sum_{n=2}^{\infty} \log(n^3+1) - \log(n^3-1) = \log\left(\frac{3}{2}\right)$$

5. Let A be an invertible $n \times n$ matrix with complex entries. Suppose that for each positive integer m, there exists a positive integer k_m and an $n \times n$ invertible matrix B_m such that $A^{k_m m} = B_m A B_m^{-1}$. Show that all eigenvalues of A are equal to 1.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A (which are allowed to repeat). Then $\lambda_1^{k_m m}, \ldots, \lambda_n^{k_m m}$ are the eigenvalues of $A^{k_m m}$, and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $B_m A B_m^{-1}$. Therefore, we have for all $m \in \mathbb{N}$ that

$$\{\lambda_1^{k_m m}, \ldots, \lambda_n^{k_m m}\} = \{\lambda_1, \ldots, \lambda_n\}.$$

So for each $1 \leq i \leq n$ and $m \in \mathbb{N}$, $\lambda_i^{k_m m} = \lambda_{a_{m,i}}$ for some $a_{m,i} \in \{1, \ldots, n\}$. By the pigeonhole principle, there exist $m, m' \in \mathbb{N}, k_m m \neq k_{m'} m'$ such that $\lambda_i^{k_m m} = \lambda_i^{k_{m'} m'}$ and thus $\lambda_i^{k_m m - k_{m'} m'} = 1$. Hence, λ_i is a root of unity for all $1 \leq i \leq n$.

Then there exists $m \in \mathbb{N}$ such that $\lambda_1^m = \cdots = \lambda_n^m = 1$. Then $\{1\} = \{\lambda_1^{k_m m}, \ldots, \lambda_n^{k_m m}\} = \{\lambda_1, \ldots, \lambda_n\}$. We are done.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function whose second derivative is continuous. Suppose that f and f'' are bounded. Show that f' is also bounded.

Assume that $|f(x)|, |f''(x)| \leq M$ for some $M > 0$ for all $x \in \mathbb{R}$. If f' is not bounded, then there exists $x_0 \in \mathbb{R}$ such that $|f'(x_0)| > N$ for some $N \in \mathbb{R}$ to be chosen later. We prove the case where $f'(x_0) > N$; the proof of the other case is similar.

Since $|f''(x)| \leq M$ for all $x \in \mathbb{R}$, by the Mean Value Theorem, we have

$$f'(x) > \frac{N}{2} \quad \text{for all } x \text{ such that} \quad x_0 - \frac{N}{2M} \leq x \leq x_0$$

since if not, then there exists $y \in [x_0, x]$ such that

$$f''(y) = \frac{f'(x_0) - f'(x)}{x - x_0} > \frac{N/2}{N/2M} = M,$$

a contradiction. Define $y_0 := x_0 - \frac{N}{2M}$. Again by the Mean Value Theorem,

$$f(x_0) - f(y_0) \geq \frac{N^2}{4M}$$

since if not, then there exists $y_0 \leq z \leq x_0$ such that

$$f'(z) = \frac{f(x_0) - f(y_0)}{x_0 - y_0} < \frac{N^2/4M}{N/2M} = N/2,$$

a contradiction. Hence one of $|f(x_0)|$ and $|f(y_0)|$ is at least $N^2/8M$. We arrive at a contradiction if we choose N so that $N^2 > 8M^2$ since we then have

$$\max\{|f(x_0)|, |f(y_0)|\} \geq \frac{N^2}{8M} > M$$