Chapter 5

Cycles and Circuits

Section 5.1 Eulerian Graphs

Probably the oldest and best known of all problems in graph theory centers on the bridges over the river Pregel in the city of Königsberg (presently called Kaliningrad in Russia). The legend says that the inhabitants of Königsberg amused themselves by trying to determine a route across each of the bridges between the two islands (*A* and *B* in Figure 5.1.1), both river banks (*C* and *D* of Figure 5.1.1) and back to their starting point using each bridge exactly one time. After many attempts, they all came to believe that such a route was not possible. In 1736, Leonhard Euler [15] published what is believed to be the first paper on graph theory, in which he investigated the Königsberg bridge problem in mathematical terms.

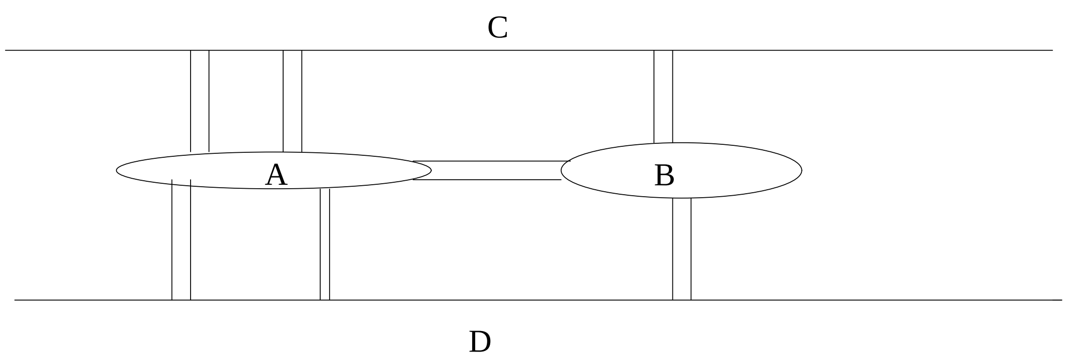


Figure 5.1.1. The bridges on the river Pregel.

The solution to the bridge problem hinges on the degrees of the vertices in the graph model for the bridges and land masses (see Figure 5.1.2). The problem seeks a circuit that contains each edge. In honor of Euler, we say a graph (or multigraph) is *eulerian* if it has a circuit containing all the edges of the graph. The circuit itself is called an *eulerian circuit.* A trail containing every edge of the graph is called an *eulerian trail.*

Several characterizations have been developed for eulerian graphs. Euler has frequently been credited with the complete equivalence of statements 1 and 2 in Theorem 5.1.1; however, he actually established only half of it ($1 \rightarrow 2$). Hierholzer [33] established the converse. The result below is a blend of the work of Euler [15], Hierholzer [33] and Veblen [46].

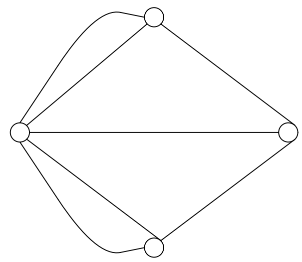


Figure 5.1.2. The multigraph of the bridges.

Theorem 5.1.1 The following statements are equivalent for a connected graph G:

1. The graph G contains an eulerian circuit.
2. Each vertex of G has even degree.
3. The edge set of G can be partitioned into cycles.

Proof. To see that (1) implies (2), let v be a vertex of G. If C is an eulerian circuit in G, then each edge of G entering v must be matched by another edge leaving v. Thus, the degree of any vertex must be even.

To establish that (2) implies (3), we use induction on the number of cycles in the graph. Clearly, the result holds if there are no cycles or there is one cycle in G. Now, assume the result holds if there are fewer than k cycles in G and suppose that G is a connected graph with all vertices having even degree and containing exactly k cycles. Delete the edges of one cycle C^* of G. Each component of the remaining graph clearly has the property that all vertices have even degree. Further, each component has fewer than k cycles, and, thus, by the induction hypothesis, their edge sets can be partitioned into cycles. The edge set of G can then be partitioned in a similar manner and along with C^* we obtain the desired partition of $E(G)$.

Finally, to see that (3) implies (1), assume that the edge set of G can be partitioned into cycles and suppose there are m such cycles. We use induction to establish that for any k $(1 \le k \le m)$, there is a circuit in G which contains each edge of k of the cycles and no other edge of G. The statement is clear if $k = 1$, since a single cycle suffices. Thus, we assume there is a circuit C which contains k of the cycles $(k < m)$ and no other edges of G. Since G is connected, at least one of the other cycles must contain a vertex of C.

Let C_1 be the circuit obtained by traversing that cycle, beginning at some common vertex v (and, hence, returning there), and then following C. Then clearly, C_1 contains the edges of $k + 1$ cycles and no other edges; hence, the result follows by induction. □

Since every graph contains an even number of vertices of odd degree, the following corollary is easily obtained.

Corollary 5.1.1 A connected graph G contains an eulerian trail if, and only if, at most two vertices of G have odd degree.

We can, however, extend Corollary 5.1.1 a little further. The proof is left to the exercises.

Corollary 5.1.2 Let $G = (V, E)$ be a connected graph with $2k$ $(k > 0)$ vertices of odd degree. Then E can be partitioned into exactly k open trails.

These ideas can certainly be extended to directed graphs. The literature on this subject contains a wide variety of terms for such digraphs. We shall simply call them eulerian digraphs. The following result is analogous to Theorem 5.1.1. Its proof is also left to the exercises.

Theorem 5.1.2 The following statements are equivalent for a connected digraph $D = (V, E)$:

1. The digraph D has a directed eulerian circuit.
2. For each vertex $v \in V$, $id\ v = od\ v$.
3. The arc set E can be partitioned into directed cycles.

Next, we wish to consider algorithms designed to produce eulerian circuits. Perhaps the oldest such algorithm is from Fleury [22]. Fleury's approach is to construct a trail C that will grow to be the desired eulerian circuit, constantly expanding the number of edges being used in C, while avoiding bridges in the graph formed by the edges not yet included in C. A bridge is selected only when no other choice remains. The postponing of the use of bridges is really the critical feature of this algorithm, and its purpose is to avoid becoming trapped in some component of $G - C$.

Algorithm 5.1.1 Fleury's Algorithm.

Input: A connected (p, q) graph $G = (V, E)$.
Output: An eulerian circuit C of G.
Method: Expand a trail C_i while avoiding bridges in $G - C_i$, until no other choice remains.

1. Choose any $v_0 \in V$ and let $C_0 = v_0$ and $i \leftarrow 0$.
2. Suppose that the trail $C_i = v_0, e_1, v_1, \ldots, e_i, v_i$ has already been chosen:
   a. At v_i, choose any edge e_{i+1} that is not on C_i and that is not a bridge of the graph $G_i = G - E(C_i)$, unless there is no other choice.
   b. Define $C_{i+1} = C_i, e_{i+1}, v_{i+1}$.
   c. Let $i \leftarrow i + 1$.
3. If $i = |E|$
   then halt since $C = C_i$ is the desired circuit;
   else go to 2.

Theorem 5.1.3 If G is eulerian, then any circuit constructed by Fleury's algorithm is eulerian.

Proof. Let G be an eulerian graph. Let $C_p = v_0, e_1, \ldots, e_p, v_p$ be the trail constructed by Fleury's algorithm. Then clearly, the final vertex v_p must have degree 0 in the graph G_p, and hence $v_p = v_0$, and C_p is a circuit.

Now, to see that C_p is the desired circuit, suppose instead that C_p is not an eulerian circuit of G. Thus, there must be edges of G not on C_p. Let S be the set of vertices of positive degree in G_p. Hence, $S \cap V(C_p)$ is nonempty since G is connected and $v_p \in \overline{S} = V - S$. Let i be the largest integer such that $v_i \in S \cap C_p$ but $v_{i+1} \in \overline{S}$. Since C_p ends in $\overline{S}$, it follows that $i < p$. From the definition of $\overline{S}$, each edge of G_i that joins S and $\overline{S}$ is on C_p; thus, the edge e_{i+1} is the only edge from S to $\overline{S}$ in the graph G_i. But then e_{i+1} is a bridge in G_i.

Suppose that e is any other edge of G_i that is incident to v_i. Then from step 2 of the algorithm, it follows that e must also be a bridge of G_i (and, hence, of the graph H_i, induced by S in G_i). Since $H_i \subseteq H_p$ (the graph induced by S in G_p), it follows that e is also a bridge in H_p. Further, since e_{i+1} is a bridge of G_i and v_i is the last vertex on C_p that is also in S, we see that $H_i = H_p$ and that $deg_{H_p}\, v = deg_{G_p}\, v$ for every vertex v of H_p. Thus, every vertex in H_p has even degree. But we know from exercise 15 in Chapter 2 that this implies that H_p contains no bridges, a contradiction. □

Can you determine the complexity of Fleury's algorithm?

Hierholzer [33] developed an algorithm that produces circuits in a graph *G* which are pairwise edge disjoint. When these circuits are put together properly, they form an eulerian circuit of *G*. This patching together of circuits hinges of course, on the circuits having a common vertex, and this fact follows from the connectivity of the graph. Once one circuit is formed, if all edges have not been used, then there must be one edge that is incident to a vertex of the circuit, and we use this edge to begin the next circuit. These circuits then share a common vertex.

Algorithm 5.1.2 Hierholzer's Algorithm.
Input: A connected graph $G = (V, E)$, each of whose vertices has even degree.
Output: An eulerian circuit *C* of *G*.
Method: Patching together of circuits.

1. Choose $v \in V$. Produce a circuit C_0 beginning with *v* by traversing at each step, any edge not yet included in the circuit. Set $i = 0$.
2. If $E(C_i) = E(G)$;
   then halt since $C = C_i$ is an eulerian circuit;
   else choose a vertex v_i on C_i that is incident to an edge not on C_i. Now build a circuit C_i^* beginning with v_i in the graph $G - E(C_i)$. (Hence, C_i^* also contains v_i.)
3. Build a circuit C_{i+1} containing the edges of C_i and C_i^* by starting at v_{i-1}, traversing C_i until reaching v_i, then traversing C_i^* completely (hence, finishing at v_i) and then completing the traversal of C_i. Now set $i \leftarrow i + 1$ and go to 2.

Tucker [45] developed an algorithm that is essentially a combination of the methods of Fleury and Hierholzer. In order to present his algorithm, we need another idea. If $e_1 = vw$ and $e_2 = vx$ are two edges of *G*, then the *splitting away of* e_1 *and* e_2 results in a new graph *H* obtained by deleting e_1 and e_2 from *G* and adding a new vertex *z* adjacent to *w* and *x* (see Figure 5.1.3). Note that *G* can be recovered from *H* by *identifying* the vertices *z* and *v*, that is, by replacing *z* and *v* with one new vertex adjacent to all the neighbors of *z* and *v*. It is clear that *H* is eulerian if, and only if, *G* is eulerian and $\{ e_1, e_2 \}$ does not form a cut set. It is also not hard to prove that *H* is connected if, and only if, *G* is connected and $\{ e_1, e_2 \}$ does not form a cut set (see exercise 6 in Chapter 5). We now present Tucker's algorithm. Repetition of the splitting away process is intended to produce the cycle partition of *E* promised in Theorem 5.1.1.

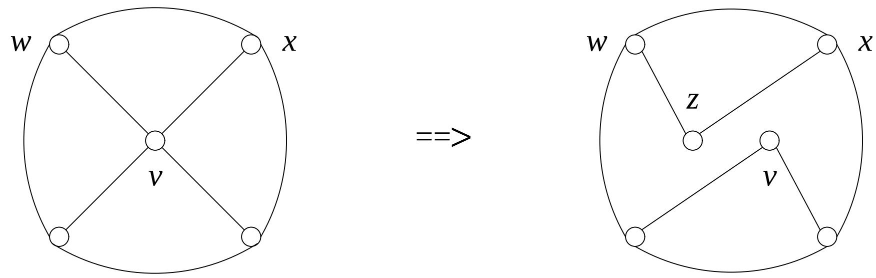


Figure 5.1.3. Splitting away of e_1 and e_2.

Algorithm 5.1.3 Tucker's Algorithm.
Input: An eulerian graph $G = (V, E)$.
Output: An eulerian circuit C of G.
Method: Split away the edges to form the desired cycle partition and then form the circuit by reassembling the cycles in a controlled manner.

1. Split away pairs of edges of G until there are only vertices of degree 2 remaining. Call the graph obtained in this manner G_1. Set $i \leftarrow 1$ and let c_i be the number of components of G_i.

2. If $c_i = 1$,
   then halt with $C = G_i$;
   else find two components T and T^* of G_i with the vertex v_i in common. Form a circuit C_{i+1}, starting at v_i and traversing T and T^*, ending again at v_i (as discussed earlier).

3. Define $G_{i+1} = G_i - \{T, T^*\} \cup C_{i+1}$. (Here, we consider the circuit C_{i+1} as being a component of G_{i+1}.) Set $T = C_{i+1}$, $i \leftarrow i + 1$, let c_i be the number of components of G_i and go to 2.

We conclude this section with the famous problem, introduced by Kwan [34] in 1960, called the *Chinese postman problem.* The problem is that the postman must traverse his mail route each day, traveling along each road and eventually returning to the post office, his starting point. Certainly, the postman desires a route that minimizes the total distance he travels.

Clearly, we can use a graph to model the mail route he must eventually cover. A solution is provided by a closed walk of minimum length that uses each edge at least once in the graph modeling the entire mail route.

It is a simple observation that if the (p, q) graph G models the postman's route, then certainly the distance the postman must travel is at least q. It is also easy to see that if the graph under consideration is a tree, then each edge must be used twice. Certainly, there is no need to use any edge three times. Hence, for the distance d in the Chinese postman problem, $q \leq d \leq 2q$. It is further clear that if the graph under consideration is an eulerian graph, then the Chinese postman problem can be solved by any of our previous eulerian algorithms. Goodman and Hedetniemi [26] noticed that the problem of finding the postman's walk is equivalent to the problem of determining the minimum number of edges that must be duplicated in order to produce an eulerian multigraph. One way to solve this problem is to consider all possible pairs of vertices of odd degree and pair them according to the minimum distances between them. Insert these edges to form the desired eulerian multigraph. We can now apply any of our algorithms for finding eulerian circuits to complete the task. However, in general this is not an efficient algorithm.

Edmonds and Johnson [13] provided a good algorithm for solving this problem, under the generalized condition that the edge distances were $w(e) \geq 0$. However, their solution is beyond the scope of this text.

Section 5.2 Adjacency Conditions for Hamiltonian Graphs

In the previous section we tried to determine a *tour* of a graph that would use each edge once and only once. It seems natural to vary this question and try to visit each vertex once and only once. In fact, a cycle (path) that contains every vertex of a graph is called a *hamiltonian cycle* (*path*).

This terminology is used in honor of Sir William Rowan Hamilton, who, in 1865, described an idea for a game in a letter to a friend. This game, called the icosian game, consisted of a wooden dodecahedron (see Figure 5.2.1) with pegs inserted at each of the twenty vertices. These pegs supposedly represented the twenty most important cities of the time. The object of the game was to mark a route (following the edges of the dodecahedron) passing through each of the cities only once and finally returning to the initial city. Hamilton sold the idea for the game. (I think his profit was the only one made on this game!)

From this unusual beginning has sprung a stream of interesting work concerning spanning cycles and paths in graphs. We cannot even begin to completely survey the extensive literature on this subject. Instead, we wish to hit some of the highlights and show some of the diverse approaches taken in what has become a very interesting and important subject.

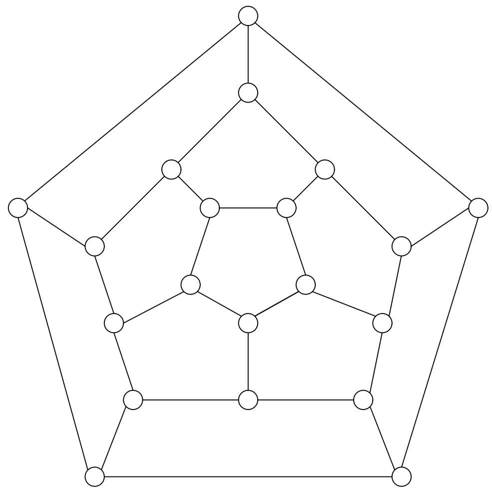


Figure 5.2.1. The graph of the dodecahedron.

Since the idea of a hamiltonian cycle "sounds" analogous to that of an eulerian circuit, one might expect that we will be able to establish some sort of corresponding theory. This, however, is definitely not the case. In fact, the hamiltonian problem is NP-complete (see [24]), and no practical characterization for hamiltonian graphs has been found. Hence, we shall not study any general algorithms for directly finding such cycles. However, there has still been a considerable amount of information discovered about hamiltonian graphs.

Perhaps the theorem that stirred the most subsequent work is that of Ore [41]. It stems from the idea that if a sufficient number of edges are present in the graph, then a hamiltonian cycle will exist. To ensure a sufficient number of edges, we try to keep the degree sum of nonadjacent pairs of vertices at a fairly high level. We can see the effect that controlling degree sums provides when we consider a vertex x of "low" degree. Since x has many nonadjacencies in the graph, the degrees of all these vertices are then forced to be "high," to ensure that the degree sum remains sufficiently large. Thus, the graph has many vertices of high degree, and so, one hopes it contains enough structure to ensure that it is hamiltonian.

Theorem 5.2.1 If G is a graph of order $p \geq 3$ such that for all pairs of distinct nonadjacent vertices x and y, $deg\ x + deg\ y \geq p$, then G is hamiltonian.

Proof. Suppose the result is not true. Then there exists a maximal nonhamiltonian graph G of order $p \geq 3$ that satisfies the conditions of the theorem. That is, G is nonhamiltonian, but for any pair of nonadjacent vertices x and y in G, the graph $G + xy$

is hamiltonian. Since $p \geq 3$, the graph G is not complete.

Now, consider $H = G + xy$. The graph H is hamiltonian and, in fact, every hamiltonian cycle in H must use the edge from x to y. Thus, there is an $x - y$ hamiltonian path P: $x = v_1, v_2, \ldots, v_p = y$ in G. Now, observe that if $v_i \in N(x)$, then $v_{i-1} \notin N(y)$, or else (see Figure 5.2.2)

$$v_1, v_i, v_{i+1}, \ldots, v_p, v_{i-1}, v_{i-2}, \ldots, v_1$$

would be a hamiltonian cycle in G. Hence, for each vertex adjacent to x, there is a vertex of $V - \{ y \}$ not adjacent to y. But then, we see that $deg\ y \leq (p - 1) - deg\ x$. That is, $deg\ x + deg\ y \leq p - 1$, a contradiction. □

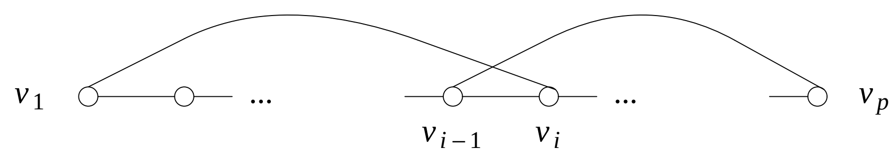


Figure 5.2.2. The path P and possible adjacencies.

We now state an immediate corollary of Ore's theorem that actually preceded it. This result is originally from Dirac [11]. A more elegant proof that inspired Ore's work was provided independently by Newman [40].

Corollary 5.2.1 If G is a graph of order $p \geq 3$ such that $\delta(G) \geq \frac{p}{2}$, then G is hamiltonian.

Following Ore's theorem, a stream of further generalizations were introduced. This line of investigation culminated in the following work of Bondy and Chvátal [4]. The next result stems from their observation that Ore's proof does not need or use the full power of the statement that each nonadjacent pair satisfies the degree sum condition.

Theorem 5.2.2 Let x and y be distinct nonadjacent vertices of a graph G of order p such that $deg\ x + deg\ y \geq p$. Then $G + xy$ is hamiltonian if, and only if, G is hamiltonian.

Proof. If G is hamiltonian, then clearly $G + xy$ is hamiltonian. Conversely, if $G + xy$ is hamiltonian but G is not, then we proceed exactly as in Ore's theorem to produce a contradiction to the degree sum condition. □

The last result inspired the following definition. The *closure* of a graph G, denoted $CL(G)$, is that graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least p until no such pair remains. The first thing we must verify is that this definition is well defined; that is, since no order of operations is specified, we must verify that we always obtain the same graph, no matter what order we use to insert these edges.

Theorem 5.2.3 If G_1 and G_2 are two graphs obtained by recursively joining pairs of nonadjacent vertices whose degree sum is at least p until no such pair remains, then $G_1 = G_2$; that is, $CL(G)$ is well defined.

Proof. Denote by $e_1, e_2, \ldots, e_i$ and $f_1, f_2, \ldots, f_j$ the sequences of edges inserted in G to form G_1 and G_2, respectively. We wish to show that each e_m is in G_2 and that each f_n is in G_1.

Let $e_k = uv$ be the first edge in the sequence $e_1, \ldots, e_i$ that is not in G_2. Consider the graph $H = G + \{ e_1, \ldots, e_{k-1} \}$. Then, from the definition of G_1 it follows that

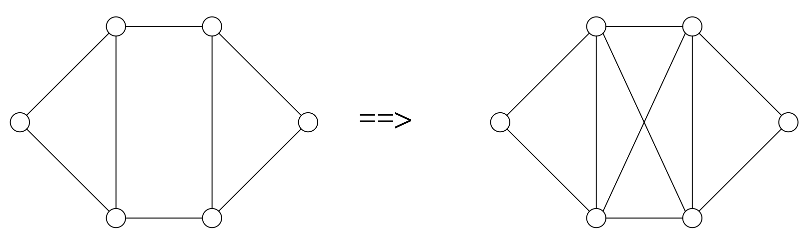
$$deg_H\ u + deg_H\ v \geq p.$$

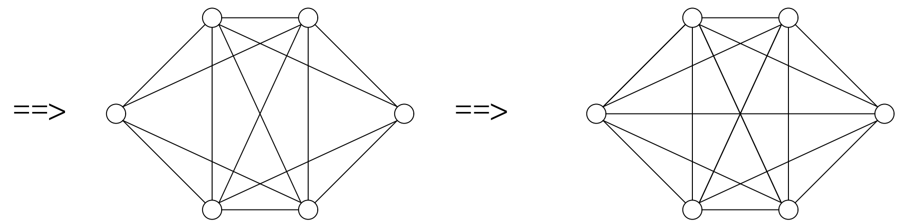
Also, by our choice of e_k, H is a subgraph of G_2. Thus,

$$deg_{G_2}\ u + deg_{G_2}\ v \geq p.$$

But this is a contradiction since u and v are nonadjacent in G_2. Similarly, each f_n is in G_1 and, hence, $G_1 = G_2$ and the closure is well defined. □

Example 5.2.1. We illustrate the construction of the closure of the graph pictured below.





Using Theorem 5.2.2, we see that the next result is immediate.

Theorem 5.2.4 A graph G is hamiltonian if, and only if, $CL(G)$ is hamiltonian.

If $CL(G) = K_p$, then it is immediate that $CL(G)$ is hamiltonian and, hence, that G is hamiltonian. This observation offers another proof of Ore's theorem and of Dirac's theorem.

In [4], a $O(V^3)$ algorithm for finding a hamiltonian cycle in a graph satisfying the degree sum condition $deg\ x + deg\ y \geq |V(G)|$ for all nonadjacent pairs of vertices x, y (call such graphs *Ore-type*) using the closure was provided. Recently, Albertson has provided a $O(V^2)$ algorithm. His approach is based on two observations taken essentially from Ore's proof technique.

Observation 1. In any Ore-type graph G, the vertices of any maximal path can be permuted to form a cycle C.

Observation 2. In any Ore-type graph G, if $x \notin V(C)$, then there is a path that contains x and all the vertices of C.

The proofs of these observations are left to the exercises. We now present Albertson's algorithm [1].

Algorithm 5.2.1 Albertson's Algorithm.
Input: An Ore-type graph G.
Output: A hamiltonian cycle in G.
Method: Use of observations 1 and 2.

1. Create a maximal path $P\colon u, \ldots, x_k, \ldots, v$.
2. Repeat while $|V(C)| \neq |V(G)|$
   If u is adjacent to v,

then set C: $u, \ldots, v, u$;
else find a k such that u is adjacent to x_{k+1} and v is adjacent to x_k. Set C: $u, x_{k+1}, x_{k+2}, \ldots, v, x_k, x_{k-1}, \ldots, x, u$.

3. Find $x \in V(G - C)$, and create P^*, a path containing x and all of C. Set P equal to a maximal path containing P^*.

4. endwhile

Next, we consider an alternate approach to ensuring a sufficient number of edges that are properly distributed to guarantee that a graph is hamiltonian. Recently, the idea of bounding the cardinality of the neighborhoods of nonadjacent vertices has been used to determine when many graph properties are present. In [18], this idea was introduced and used to determine a sufficient condition for a graph to be hamiltonian. If $G \neq K_p$, no connectivity condition can be forced by bounding the cardinality of neighborhood unions of nonadjacent vertices. (That is, once we have avoided the pathological case in which the neighborhood condition is satisfied vacuously, then no connectivity condition can be forced by the neighborhood condition.) For example, consider the graph $K_r \cup K_r$. Thus, we must assume some minimal connectivity condition.

Theorem 5.2.5 If G is a 2-connected graph such that for every pair of nonadjacent vertices x and y,

$$|N(x) \cup N(y)| \geq \frac{2p - 1}{3},$$

then G is hamiltonian.

Can you find an example of a graph G which Theorem 5.2.5 shows is hamiltonian, but for which $CL(G) = G$?

One can consider Dirac's theorem as providing a bound on the cardinality of the neighborhood of a single vertex, and the last result provides a bound on the cardinality of the neighborhood of a pair of nonadjacent vertices. In [18] it was conjectured that even higher connectivity conditions would allow expansion of the set of nonadjacent vertices to those with more than two vertices. Fraisse [23] verified that this conjecture was indeed true. We consider $N(S)$ to be the set of neighbors of the vertices in the set S. For convenience, we define $[x, y]$ to mean the segment of a path or cycle, following some orientation, from the vertex x to the vertex y (including both x and y). If we do not wish to include an end vertex we will use the notation (x, y), or a mixture of the two notations where appropriate.

Theorem 5.2.6 Let G be a k-connected graph of order $n \geq 3$. Suppose there exists some $t \leq k$ such that for every set S of t mutually nonadjacent vertices, $|N(S)| > \frac{t(n-1)}{t+1}$; then G is hamiltonian.

Proof. Suppose G is not hamiltonian and let C be a cycle of maximum length, say m, in G. Let x_0 be a vertex of G off C and fix the following ordering of C: $v_1, v_2, \ldots, v_m$. Using Menger's theorem (see exercise 14 in Chapter 2), we find that there exist t paths from x_0 to C that are disjoint except at x_0. Let $\{ w_1, w_2, \ldots, w_t \}$ denote the end vertices of these paths on C and, further, assume they occur in this order according to our ordering of C. Also, denote by x_i the successor of w_i with respect to this ordering along C and let $X = \{ x_i \mid 0 \leq i \leq t \}$. It is clear that X is a set of $t + 1$ mutually nonadjacent vertices or else a cycle longer than C would exist in G. Our strategy is to produce a subset of X that does not satisfy the hypothesis of the theorem.

We now partition $V(G)$ into the following sets:

$$A = V(G) - N(X)$$

(A is the set of vertices that are not adjacent to any vertex in X);

$$B = \{v \in V(G) \mid \text{there is a unique } i \text{ such that } vx_i \in E(G)\},$$

(B is the set of vertices u such that there is a subset Y of t vertices of X with $u \notin N(Y)$); and

$$D = V(G) - (A \cup B).$$

Further, let $A_C = A \cap V(C)$ and $A_R = A - V(C)$.

Now, $X \subseteq A$ and, therefore, $x_0 \in A_R$, or else a cycle longer than C results. Further, two vertices of X have no common neighbors in $V(G) - V(C)$, or again a longer cycle results.

Claim: There are at most t vertices of D between any two consecutive vertices of A on C.

To verify this claim, suppose that a_1 and a_2 are two consecutive vertices of A on C. Since $X \subseteq A$, we may assume that a_1 and a_2 belong to some segment of C of the form $[x_i, x_j]$ (where $0 < i < j \leq t$) or $[x_t, x_1]$. Let $v \in N(x_j)$ and (letting v^+ be the successor of v on C) let $v^+ \in N(x_k)$ ($x_k \in (x_j, x_i)$) where v and v^+ are in the segment of vertices $[a_1, a_2]$. Then, we see that x_j is not x_0 or a longer cycle would exist. In fact, if $x_0 v \in E(G)$, then $v^+ x_k \notin E(G)$ for any k, or a cycle longer than C would exist (see Figure 5.2.3). Furthermore, the vertices w_i, w_j, and w_k cannot occur in that order

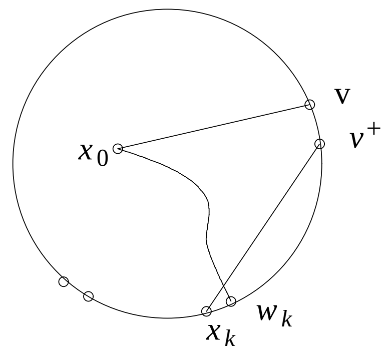


Figure 5.2.3. Extending the cycle through x_0.

along C (according to its ordering), because again a cycle longer than C would exist (see Figure 5.2.4). For example

$$x_0, \ldots, w_k, \ldots, x_j, v, \ldots, x_k, v^+, \ldots, w_j, x_0$$

is such an extension. Therefore,

$$N(x_i), N(x_{i-1}), N(x_{i-2}), \ldots, N(x_1), N(x_t), \ldots, N(x_{i+1}), N(x_0)$$

form consecutive segments of vertices in the segment $[a_1, a_2]$ (which is possibly empty). Further, these segments have at most their end vertices in common. There are $t + 1$ segments; hence, there are at most t elements of D between a_1 and a_2. Thus, our claim is verified.

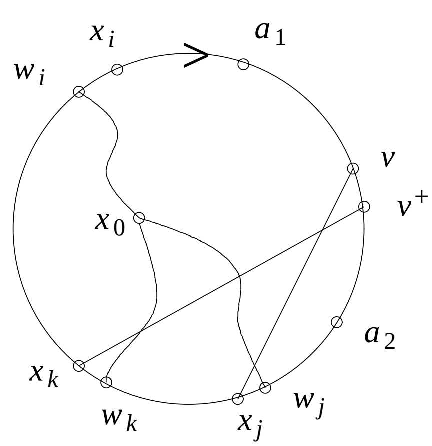


Figure 5.2.4. The ordering of w_i, w_j, w_k.

To complete the proof of the result, let x_j, $0 \le j \le t$ be chosen with the maximum number s of neighbors in B. Then $s \ge |B|/(t+1)$. Also, the number of vertices of B in $V(G) - V(C)$ equals $n - m - |A_R|$ (since two vertices of X have no common neighbors in $V(G) - V(C)$). From our claim, the number of vertices of B on C is at least $m - (t+1)\,|A_C|$. Thus,

$$|B| = |B \cap C| + |B - C|$$

$$\ge n - m - |A_R| + m - (t+1)\,|A_C|$$

$$= n - |A_R| - (t+1)\,|A_C|$$

and

$$s \ge \frac{|B|}{t+1} \ge \frac{n}{t+1} - \frac{|A_R|}{t+1} - |A_C|.$$

Since $X - \{x_j\}$ is a set of t mutually nonadjacent vertices and since $|A_R| \ge 1$,

$$|N(X - x_j)| = n - |A| - s = n - |A_R| - |A_C| - s$$

$$\le \frac{t(n - |A_R|)}{t+1} \le \frac{t(n-1)}{t+1}.$$

Hence, a contradiction is reached and the result is proved. □

Example 5.2.2. The best known examples to show the sharpness of the bound in the previous result are obtained by considering $t+1$ copies of the complete graph K_r which are identified on a set of t vertices. The Petersen graph (see Figure 6.3.2) is another example. However, these examples all leave some doubt as to the best possible lower bound.

The graph $tK_r + K_t$ satisfies the hypothesis of Fraisse's Theorem, but its n-closure is itself. Thus, Theorem 5.2.6 is independent of the closure (Theorem 5.2.2).

It is interesting to note that the type of vertex set used can play a role here. In the last theorem independent sets of vertices were considered. However, something a bit different happens if we consider arbitrary sets of vertices.

Theorem 5.2.7 [17] If G is a 2-connected graph of sufficiently large order n and if for each pair of arbitrary vertices $S = \{a, b\}$ we have that

$$|N(S)| \geq \frac{n}{2},$$

then G is hamiltonian.

It is known that $n > 10$ suffices in Theorem 5.2.7. The Petersen graph is a small order (10) counterexample.

It is natural now to ask what happens if the set of vertices induces a clique? What properties in general are important to the selection of the set? Some of these questions have been considered, but the last and most important of them remains open. The interested reader should see [28] and [35] for more information.

Section 5.3 Related Hamiltonian-like Properties

In this section we wish to investigate several properties closely related to that of being hamiltonian. Some of these are stronger properties, in the sense that the graphs having these properties are also hamiltonian, while others are weaker. It is not surprising that truly applicable characterizations of these properties are not known. In some cases, very little is actually known at all about the classes of graphs we will describe.

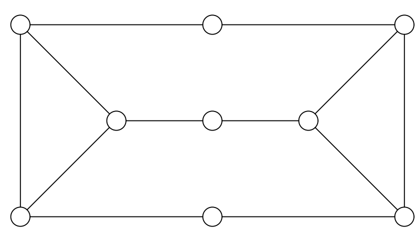


Figure 5.3.1. A homogeneously traceable nonhamiltonian graph.

To begin with, we say a graph is *traceable* if it contains a hamiltonian path. Clearly, every hamiltonian graph is also traceable, and the graphs P_n show that the converse of this statement does not hold. There are two other rather elusive classes of graphs that essentially lie between the hamiltonian and traceable classes. A graph G is *homogeneously traceable* if there is a hamiltonian path beginning at every vertex of G, while G is *hypohamiltonian* if G is not hamiltonian but $G - v$ is hamiltonian for every vertex v of G. It is easy to see that every hypohamiltonian graph is also homogeneously traceable. The graph of Figure 5.3.1 is homogeneously traceable, but not hypohamiltonian.

Skupien introduced homogeneously traceable graphs in [44], and the existence of homogeneously traceable nonhamiltonian graphs for all orders $p \geq 9$ was shown in [6]. In [32], Herz, Gaudin and Rossi showed that hypohamiltonian graphs exist, while in [31], Herz, Duby and Vigué showed that there is no hypohamiltonian graph of order 11 or 12. Lindgren [37] and Sousselier (see [31]) independently showed that infinitely many hypohamiltonian graphs exist. Both of these classes have remained elusive in the sense that very few results are known about them, especially in view of the vast number of results known about hamiltonian graphs. Some simple facts will be explored in the exercises.

More success has been had with properties stronger than that of being hamiltonian. We say a graph G is *hamiltonian connected* if every two vertices of G are joined by a hamiltonian path. Clearly, every hamiltonian connected graph of order at least 3 is hamiltonian; the graphs C_n $(n \geq 4)$ show that the converse is not true. We can use a generalization of the idea of the closure to obtain a sufficient condition for a graph to be hamiltonian connected. For a (p, q) graph G, let the $(p + 1)$-closure denoted $CL_{p+1}(G)$, be that graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least $p + 1$. We obtain the following analog of Theorem 5.2.4, also from Bondy and Chvátal [4].

Theorem 5.3.1 Let G be a graph of order p. If $CL_{p+1}(G)$ is complete, then G is hamiltonian connected.

Proof. If $p = 1$, the result is clear. Thus, assume that $p \geq 2$ and that G is a graph of order p such that $CL_{p+1}(G) = K_p$. Let x and y be two vertices of G. We show that x and y are joined by a hamiltonian path in G, and since they are an arbitrary pair of vertices, the result will follow.

Let H be the graph obtained from G by inserting a new vertex w and the edges wx and wy. Then H has order $p^* = p + 1$.

Since $deg_H\ u + deg_H\ v \geq p^*$ for all nonadjacent vertices u and v in G, it follows that $< V(G) >$ in $CL(H)$ must be isomorphic to K_p. Thus, if $u \in V(G)$ and $uw \notin E(H)$, then $deg_{CL(H)}\ u + deg_{CL(H)} w \geq p^*$, and, hence, $CL(H)$ is isomorphic to K_{p^*}. Then by Ore's theorem, H is hamiltonian. But any hamiltonian cycle in H must contain the edges wx and wy, and, thus, x and y are end vertices of a hamiltonian path in G. □

The next two corollaries are analogs of the theorems of Ore and Dirac and are immediate from our previous result, although they can also be proved directly.

Corollary 5.3.1 If G is a graph of order p such that for every pair of distinct nonadjacent vertices x and y in G, $deg\ x + deg\ y \geq p + 1$, then G is hamiltonian connected.

Corollary 5.3.2 If G is a graph of order p such that $deg\ x \geq \frac{p+1}{2}$ for every vertex x, then G is hamiltonian connected.

Yet another hamiltonian-like property is the following: A connected graph $G = (V, E)$ is said to be *panconnected* if for each pair of distinct vertices x and y, there exists an $x - y$ path of length l, for each l satisfying $d(x, y) \leq l \leq |V| - 1$. If a graph G is panconnected, it is clearly hamiltonian connected and, thus, hamiltonian. It is also easy to see that there are hamiltonian graphs that are not panconnected (for example, cycles). Can you find a graph that is hamiltonian connected but not panconnected? In a result similar to Dirac's theorem, Williamson [47] provided a sufficient condition for a graph to be panconnected in terms of minimum degree.

Theorem 5.3.2 If G is a graph of order $p \geq 4$ such that for every vertex $x \in V(G)$, $deg\ x \geq \frac{p+2}{2}$, then G is panconnected.

Our final properties are somewhat related. We say a graph G of order p is *pancyclic* if it contains a cycle of every length l, $(3 \leq l \leq p)$. We say G is *vertex pancyclic* if each vertex of G lies on a cycle of each length l, $(3 \leq l \leq p)$. Clearly, every pancyclic and every vertex pancyclic graph is hamiltonian. It is also clear that every vertex pancyclic graph is pancyclic. Can you find a graph that is pancyclic, but not vertex pancyclic? A sufficient condition for a graph to be pancyclic was provided by Bondy [3].

Theorem 5.3.3 If G is a hamiltonian (p, q) graph with $q \geq \frac{p^2}{4}$, then either G is pancyclic or p is even and G is isomorphic to $K_{p/2,\ p/2}$.

Many hamiltonian results, similar to those we have studied, exist for digraphs. We explore some of these in the exercises.

Section 5.4 Forbidden Subgraphs

In this section we consider another approach that has seen considerable study. Goodman and Hedetniemi [27] noticed that forbidding particular induced subgraphs offered a new type of hamiltonian result. If $F = \{H_1, \ldots, H_k\}$ is a family of graphs, we say that G *is F-free* if G does not contain any of the graphs in the set F as an induced subgraph. If F is a single graph H we simply say that G is H-free. We define $Z_1 = K_{1,3} + e$.

Theorem 5.4.1 ([27]) If G is a 2-connected $\{K_{1,3}, Z_1\}$-free graph, then G is hamiltonian.

Proof. Suppose G is not hamiltonian. Since G is 2-connected, it contains cycles. Let $C : x_1, x_2, \ldots, x_n, x_1$ be a cycle in G of maximum length. Then, there must exist a vertex x not on C that is adjacent to a vertex, say x_i, of C. Then, $< x, x_i, x_{i-1}, x_{i+1} >$ induces either a $K_{1,3}$ or a Z_1, unless at least two of the edges xx_{i-1}, xx_{i+1}, or $x_{i-1}x_{i+1}$ are present in G. But then no matter how these edges are selected, one of xx_{i-1} or xx_{i+1} must be present, and a cycle longer than C is produced, a contradiction to our choice of C. Thus, G is hamiltonian. □

Theorem 5.4.1 inspired several others. One of the earliest and strongest of these is now stated. The graph N (known as the net) is seen in Figure 5.4.1.

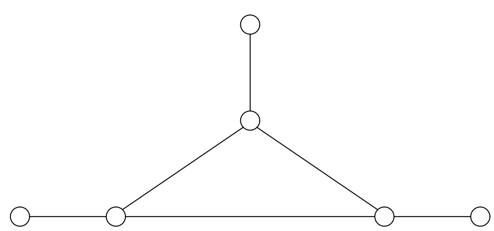


Figure 5.4.1. The graph N.

Theorem 5.4.2 ([12]) If G is

1. connected and $\{K_{1,3}, N\}$-free, then G is traceable.
2. 2-connected and $\{K_{1,3}, N\}$-free, then G is hamiltonian.

Note that if H is an induced subgraph of some graph S and G is H-free, then G is also S-free, for if G contained an induced S, it clearly would contain an induced H as well. Thus, we get the following corollary to Theorem 5.4.2. The graph B is shown in Figure 5.4.2.

Corollary 5.4.1 If G is a 2-connected graph that is $\{R, S\}$-free where $R = K_{1,3}$ and S is any one of N, C_3, Z_1, B, P_4, P_3, K_2 or K_1, then G is hamiltonian.

In the list of graphs given in Corollary 5.4.1, we note that if P_3 is forbidden, the graph must clearly be complete, thus it has any hamiltonian property you wish. Further, if the induced subgraphs of P_3 are forbidden, namely K_2 or K_1, we either have that G must be K_1 or G is empty. In either case we arrive at trivial cases that satisfy all hamiltonian properties. For this reason in what remains we will ignore P_3 and its induced subgraphs as simply being trivial and therefore "uninteresting".

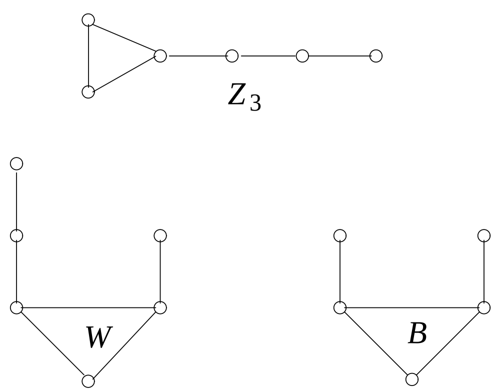


Figure 5.4.2. Common forbidden subgraphs.

After Theorem 5.4.2 was announced, the search for other familes of forbidden subgraphs began in earnest. Over the course of the next decade a variety of such results were discovered. We summarize the most important of these for our purposes in the next result. See Figure 5.4.2 for the graphs W and Z_3, where by Z_i we mean a triangle with a path containing exactly i edges from one of its vertices.

Theorem 5.4.3 If G is a 2-connected graph and G is

i. ([5]) $\{K_{1,3}, P_6\}$-free, or

ii. ([2]) $\{K_{1,\,3},\ W\}$-free, or

iii. ([20]) $\{K_{1,\,3},\ Z_3\}$-free and of order $p \geq 10$,

then G is hamiltonian.

Recently, a characterization of all pairs of graphs that, when forbidden, imply a 2-connected graph is hamiltonian was given in [16].

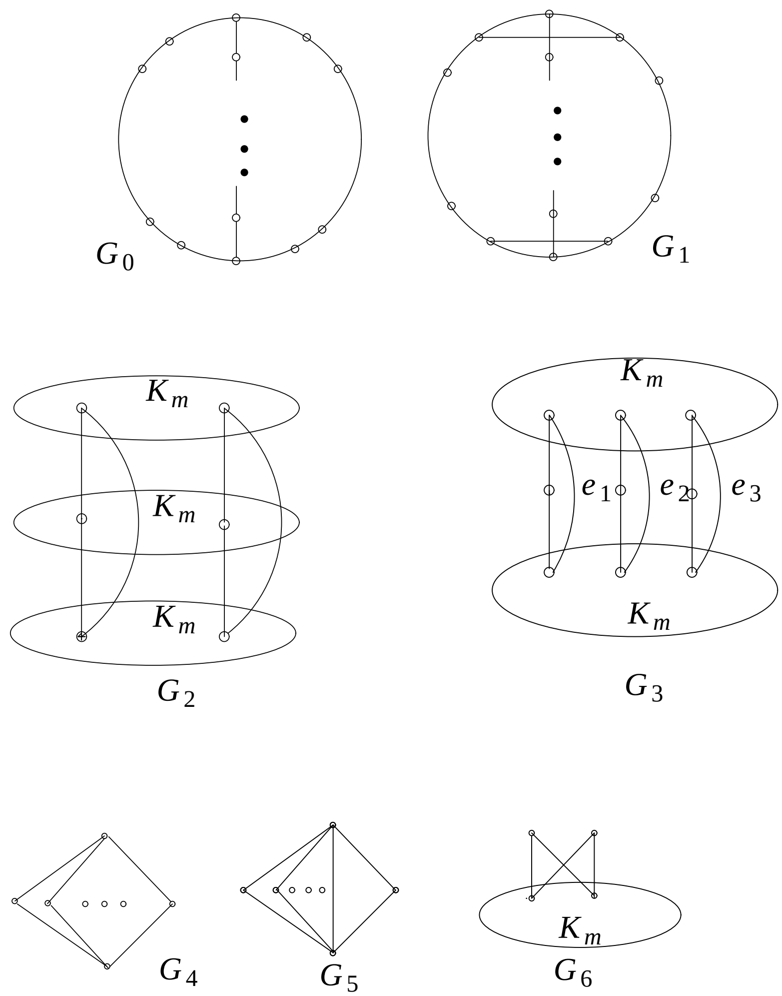


Figure 5.4.3. 2-Connected, nonhamiltonian graphs.

Theorem 5.4.4 Let R and S be connected graphs ($R,\ S \neq P_3$) and G a 2-connected graph of order $n \geq 10$. Then G is (R, S)-free implies G is hamiltonian if, and only if, $R = K_{1,3}$ and S is one of the graphs C_3, P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N or W.

Proof: That each of the pairs implies G is hamiltonian follows from Theorems 5.4.2 and 5.4.3 and our earlier remarks about induced subgraphs of forbidden graphs.

Now consider the graphs $G_0, \ldots, G_6$ of Figure 5.4.3. Each is 2-connected and nonhamiltonian (and really represents an infinite family of such graphs). Without loss of generality assume that R is a subgraph of G_1.

Case 1: Suppose that R contains an induced P_4.

Since G_4, G_5, and G_6 are all P_4-free, then S must be an induced subgraph of each of them. But if S is an induced subgraph of G_4, then either S is a star or S contains an induced C_4. However, G_5 is C_4-free, hence S must be a star. Since the only induced star in G_6 is $K_{1,3}$, we have that $S = K_{1,3}$.

Case 2: Suppose that R does not contain an induced P_4.

Then, using G_0 we see immediately that R must be a tree containing at most one vertex of degree 3 and since R contains no induced P_4, we see that $R = K_{1,3}$. Thus, for the remainder of the proof we assume without loss of generality that $R = K_{1,3}$.

Now, S must be an induced subgraph of G_1, G_2, and G_3 (each of which is claw-free). The fact that S is an induced subgraph of G_1 implies that S is a path or S is K_3, possibly with a path off each of its vertices. Suppose that S is a path. Since S is an induced subgraph of G_3 which is P_7-free, we see that if S is a path, it is one of P_4, P_5 or P_6.

We now assume that S contains a K_3, possibly with a path off each of its vertices. Note that G_3 is Z_4-free. Further, any triangle in G_2 with a 3-path off one of its vertices can have no paths off its other vertices (leaving Z_3, Z_2, Z_1, and K_3). Again examining G_2 we see it contains no triangle with a path of length 2 from one of its vertices and a path of length 1 from the other two vertices. Next, the graph G^*_3 obtained by deleting the three edges e_1, e_2 and e_3 from G_3 is still claw-free and contains no induced K_3 with a 2-path from two vertices. (leaving B or W). The only remaining possibility is a path of length 1 off each of the vertices of K_3, that is, the graph N. □

Note that similar results are known for traceable graphs as well as for several other hamiltonian type properties. These will be explored in more detail in the exercises.

Section 5.5 Other Types of Hamiltonian Results

In this section we wish to consider several other types of results concerning hamiltonian and hamiltonian-like graphs. These use conditions that are often very different from those we have seen thus far. We begin with an investigation of the powers of a graph.

The *nth power* G^n of a connected graph G is that graph with $V(G^n) = V(G)$ and in which *uv* is an edge of G^n if, and only if, $1 \leq d_G(u, v) \leq n$. The graphs G^2 and G^3 are called the *square* and *cube* of G, respectively. Figure 5.5.1 shows the *subdivision graph* of the graph $K_{1, 3}$, that is, the graph $S(K_{1, 3})$ obtained by *subdividing* each edge of $K_{1, 3}$. The graph $S(K_{1, 3})$ is formed from $K_{1, 3}$ when each edge $e = xy$ is removed and a new vertex w is inserted along with the edges wx and wy. Figure 5.5.1 also shows the square of $S(K_{1, 3})$.

Since higher powers of a graph G tend to contain more edges than G itself, it is reasonable to ask if these powers will eventually become hamiltonian, even if G is not. Nash-Williams and Plummer conjectured that this was indeed the case for the squares of 2-connected graphs. In the now classic paper [21], Fleischner verified that this was indeed the case.

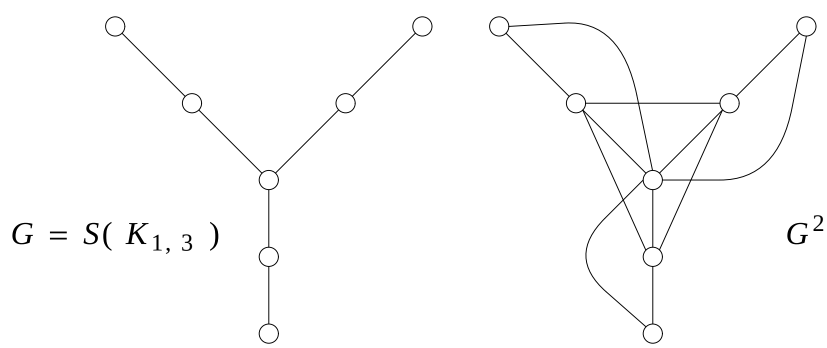


Figure 5.5.1. The graph $S(K_{1, 3})$ and its square.

Theorem 5.5.1 Let G be a 2-connected graph. Then G^2 is hamiltonian.

The proof of Fleischner's theorem is beyond the scope of most texts and, hence, will not be presented. However, this work opened the door for others to investigate properties in powers of graphs. In [7], Fleischner's result was strengthened to show that the square of every 2-connected graph is actually hamiltonian connected (see the exercises).

Let us switch attention now to hamiltonian properties of line graphs. In [30], Harary and Nash-Williams characterized when the line graph $L(G)$ of a graph G is hamiltonian. In order to do this, we define a *dominating circuit* C of a graph G to be a circuit with the property that every edge of G is incident to a vertex of C.

Theorem 5.5.2 Let G be a graph without isolated vertices. Then $L(G)$ is hamiltonian if, and only if, G is isomorphic to $K_{1,\, n}$ $(n \geq 3)$ or G contains a dominating circuit.

The major impact of Theorem 5.5.2 has been its use in proving other results.

In [29], the idea of dominating circuits is used to help establish a result that combines ideas from throughout this section. Suppose that H is a connected graph and that $L(H)$ is not hamiltonian. Therefore, H must not contain a dominating circuit. Among all circuits of H, let C be one with the maximum number of vertices. Let $V = V(C)$ and let $U = V(H) - V$. Since C is not a dominating circuit, $< U >$ is not an empty graph. Thus, since H is connected, there exists at least one edge from U to V in H. With this in mind, we present the following lemma [29].

Lemma 5.5.1 No $U - V$ edge of H lies on a triangle T of H.

Proof. Suppose that the result failed to hold; then one of the following cases would have to occur.

Case 1: Exactly one vertex of the triangle T lies on C and, thus, a circuit longer than C would exist, a contradiction.

Case 2: Exactly two vertices of the triangle T lie on C and they are consecutive on C (say they are joined by e). Then deleting e and inserting the other two edges of T creates a circuit longer than C, again a contradiction.

Case 3: Exactly two vertices of T lie on C, but these two vertices are not consecutive. Then a longer circuit can be found using all of C and all of T, once again producing a contradiction.

This completes the cases, and, thus, the lemma follows. □

Using this lemma, we can obtain a result concerning hamiltonian properties in the second iterated line graph of a graph [29].

Theorem 5.5.3 Let G be a connected graph of order $p \geq 3$ which does not contain a vertex cutset consisting only of vertices of degree 2. Then $L^2(G)$ (that is, $L(L(G))$) is hamiltonian.

Proof. Let $H = L(G)$ and suppose that $L(H) = L^2(G)$ is not hamiltonian. Then, as above, there exists a partition of $V(H)$ as $V \cup U$ such that (by the lemma) no $U - V$ edge in H lies on a triangle. Let $e_1 f_1, \ldots, e_r f_r$ ($r \geq 1$) be the $U - V$ edges in H and let v_i be the common end vertex of e_i and f_i in G ($1 \leq i \leq r$). Since $e_i f_i$ does not lie in a triangle of H, $deg_G\ v_i = 2$, ($1 \leq i \leq r$). Further, since the graph

$$H - \{ e_i f_i \mid 1 \leq i \leq r \}$$

is disconnected, then $G - \{ v_i \mid 1 \leq i \leq r \}$ is also disconnected. Therefore, we reach a contradiction since $\{ v_i \mid 1 \leq i \leq r \}$ is a cut set of G consisting entirely of vertices of degree 2. Thus, the result follows. □

With the aid of Theorem 5.5.3, the following corollary, originally from Chartrand and Wall [8], is immediate.

Corollary 5.5.1 If G is a connected graph such that $\delta(G) \geq 3$, then $L^2(G)$ is hamiltonian.

Section 5.6 The Traveling Salesman Problem

Consider the dilemma of a traveling salesman. He must visit each city in his region and return to his home office on a regular basis. He seeks the route that allows him to visit each city at least once (exactly once would be even better) and return home, with the added property that this route also covers the least distance. Clearly, a weighted graph can be used to model the possible routes. Because we can insert edges with infinite distances, we can also restrict our attention to complete graphs.

Unfortunately for the traveling salesman, this problem is NP-complete (see, for example, [24]). Thus, efforts have concentrated on approximation algorithms and the use of heuristics to try to make some gains. The heuristic one usually thinks of first, the *nearest neighbor* approach, begins with a single vertex, adds the edge of minimum distance, and continues by trying to build from either end of this path by repeatedly taking the nearest neighbor. Unfortunately, to close the path into a cycle is often very expensive, and we can also ignore many short edges this way. Thus, this approach is very unreliable.

To try to overcome this difficulty, you might try beginning with some short cycle and expand this cycle by inserting the vertex that causes the cycle length to increase the least. This technique is called the *shortest insertion heuristic*. Once again, difficulties can arise. Part of the problem stems from the fact that an arbitrarily weighted graph need not satisfy the "reasonable rules" of distance. That is, the triangle inequality need not hold. However, if the triangle inequality does hold, some progress can be made. In the algorithm that follows, we assume both a single vertex and a K_2 are (degenerate) cycles.

Algorithm 5.6.1 The Shortest Insertion Algorithm
Input: A weighted graph $G = (V, E)$ satisfying the triangle inequality.
Output: A hamiltonian cycle C that approximates the salesman cycle.

1. Select any vertex and consider it a 1-cycle C_1 of G. Set $i \leftarrow 1$.
2. If $i = p$, then halt since $C = C_p$ is the desired cycle;
   else if C_i has been selected $(1 \le i \le p)$, then find a vertex v_i not on C_i that is closest to a consecutive pair of vertices w_i and w_{i+1} of C_i.
3. Let C_{i+1} be formed by inserting v_i between w_i and w_{i+1} on C_i and go to step 2.

The type of heuristics we have just considered constructs a tour whose value is hoped to be close to optimal. These are known as *tour construction heuristics*. Other heuristics take a tour and try to improve it (known as *tour improvement heuristics*). One of the most commonly used improvement procedures is that of edge exchanges.

In general, *r* edges in a tour are exchanged for *r* edges not in the tour, as long as the result remains a tour and the length of the new tour is less than the length of the old tour. Exchange procedures have come to be called *r-opt procedures*, where *r* is the number of edges exchanged at each iteration. Figure 5.6.1 illustrates the exchange for 2-opt and 3-opt.

In an *r*-opt algorithm, all feasible exchanges of *r* edges are tested until there is no exchange that improves the current situation. The solution is than said to be *r*-optimal (see [36]). In general, the larger the value of *r*, the more likely it is that the final tour is optimal. However, the number of operations necessary to test all possible *r* exchanges increases dramatically as the number of cities increase. For this reason, $r = 2$ and $r = 3$ are the most commonly used values.

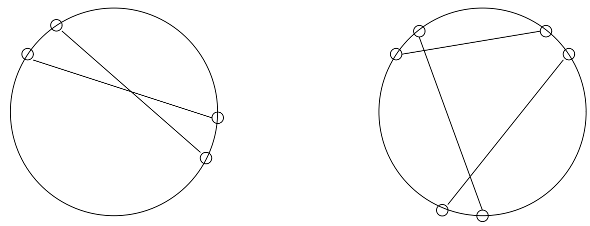


Figure 5.6.1. The 2-opt and 3-opt exchanges

Section 5.7 Short Cycles and Girth

In this section we explore a few results about nonhamiltonian cycles in graphs. Many of the same techniques that aided us in finding spanning cycles can be of use here. We begin with a result from [4].

Theorem 5.7.1 If G is a graph of order $p \geq 3$ and $5 \leq t \leq p$, then the graph G contains a C_t if, and only if, for every pair of nonadjacent vertices u and v such that $deg\ u + deg\ v \geq 2p - t$, $G + uv$ contains a C_t.

Proof. If G contains C_t, then clearly so does $G + uv$.

To prove the converse, suppose that $G + uv$ contains a C_t, but G does not. Then there exists a path P of length $t - 1$ in G joining u and v. Let $U = V(G) - V(P)$. Then it is clear that $|U| = p - t$. If u and v are each adjacent to all of U, then that still accounts for only $2(p - t)$ adjacencies. Thus, they must have at least t adjacencies on P. But now, we simply apply the technique used in the proof of Ore's theorem to produce a C_t using the vertices of P. □

We may also use neighborhoods to obtain results about nonspanning cycles.

Theorem 5.7.2 ([19]) If G is 2-connected of order p satisfying the property that for every pair of nonadjacent vertices u and v

$$|N(u) \cup N(v)| \geq s,$$

then G contains a cycle of length at least $s + 2$ or (if $p < s + 2$) G is complete.

Proof If $p < s + 2$, then the neighborhood condition implies that G must be complete.

Now, assume that $p \geq s + 2$. Let G be a 2-connected graph of order p satisfying the neighborhood condition. Let $x_1, x_2, \ldots, x_t$ be the vertices in order along a path P of maximum length in G. Let x_i and x_j (possibly $i = j$) be vertices on P that are adjacent to x_1, and suppose that, among all possible longest paths P and vertices x_i, we have chosen ones that maximize i. Then x_1 and x_{j-1} are adjacent only to vertices in

$$W = \{ x_1, \ldots, x_i \}$$

(since clearly x_{j-1} is an end vertex of a path having the same length as P). Suppose $i \leq s + 1$. Then by the neighborhood condition, we see that x_1 and x_{j-1} must be adjacent. Taking j to be $i, i - 1, \cdots$ in turn, we see that x_1 is adjacent to each of $x_2, \ldots, x_i$. All vertices $x_1, \ldots, x_{i-1}$ are, thus, end vertices of paths of the same length as P, and so they are adjacent only to vertices of W, which contradicts the 2-connectedness of G. Thus, $i \geq s + 2$, and G contains a cycle of length $s + 2$ or more. □

We now turn our attention specifically to the shortest cycle in a graph. The *girth* of a graph G, denoted $g(G)$, is the length of a shortest cycle in G. We can bound the order of G from below using the girth and the minimum degree of G. Let $f(g, \delta)$ be the minimum order of a graph G with $g(G) \geq g$ and $\delta(G) \geq \delta$. It is not immediately obvious that there exist graphs satisfying these restrictions. We shall provide an upper bound on $f(g, \delta)$ (and, hence, prove the existence of such graphs) in Chapter 9 using probabilistic techniques. For now, we shall accept their existence and try to learn more about them. One can obtain an easy lower bound on $f(g, \delta)$.

Theorem 5.7.3 For $\delta, g \geq 3$,

$$f(g, \delta) \geq \begin{cases} \dfrac{\delta(\delta - 1)^d - 2}{\delta - 2} & \text{if } g = 2d + 1, \\[2ex] \dfrac{(\delta - 1)^d - 1}{\delta - 2} & \text{if } g = 2d. \end{cases} \tag{5.7.1}$$

Proof. If g is odd, say $g = 2d + 1$, note that the subgraph induced by those vertices within a distance d of a vertex x must be a tree, since any cycle within this graph would have length less than g. Since there are at least $\delta(\delta - 1)^j$ vertices at distance $j + 1$ from x, we see that

$$f(g,\ \delta) \geq 1 + \delta \sum_{j=0}^{d} (\delta - 1)^j.$$

Since this is a geometric series, the desired result follows.

Now, suppose that g is even, say $g = 2d$. Then consider the vertices within a distance *d* of the edge *xy* and proceed in a manner similar to the odd case to obtain the bound. □

In certain cases we can find the exact value of $f(g,\ \delta)$, especially when the graph is δ-regular. For example, the 2-regular graphs of girth *g* and minimum order *p* are simply the cycles C_p. It is also easy to see that the *r*-regular graphs of girth $g = 3$ and minimum order are simply the complete graphs K_{r+1}. When $g = 4$ and the graph is *r*-regular, then the complete bipartite graphs $K_{\delta,\ \delta}$ have minimum order and these conditions. It has become common to call *r*-regular graphs of girth *g* and minimum order the $(g,\ r)$-*cages*. It turns out that $(g,\ r)$-cages are difficult to find. The first really interesting case is the $(5,\ 3)$-cage; that is, we want a 3-regular graph of girth 5 and minimum possible order. Using our tree structure (from the proof of Theorem 5.7.3), we see that such a graph has at least 10 vertices. The graph of Figure 5.7.1 has exactly 10 vertices and satisfies the other conditions as well (a fact you must convince yourself of). The interesting point here is that this graph is isomorphic to the Petersen graph and is the unique $(5,\ 3)$-cage (see the exercises).

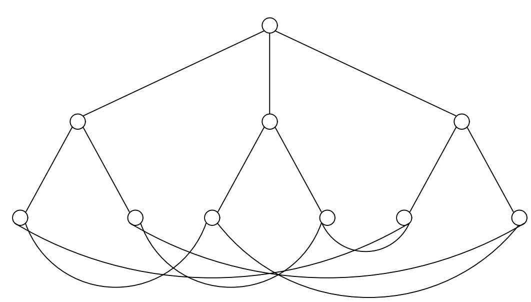


Figure 5.7.1. The $(5,\ 3)$-cage.

We next state a result summarizing some of the best known work on bounding $f(g,\ \delta)$.

Theorem 5.7.4

1. For $\delta \geq 2$, $f(4,\ \delta) = 2\delta$.

2. For $\delta \geq 2$, $f(g, \delta) \geq \delta^2 + 1$. Furthermore, for $\delta \neq 57$, equality holds if, and only if, $\delta = 2$, 3 or 7.
3. When $\delta = p^m + 1$, for some prime *p* and positive integer *m*, then

$$f(6, \delta) = \frac{2(\delta - 1)^3 - 2}{\delta - 2}.$$

We conclude with a table of orders for the known cages (Table 5.7.1).

r/g	5	6	7	8	9	12
3	10	14	24	30		126
4	19	26				
5	30	40	50			

Table 5.7.1 Known orders for cages.

Section 5.8 Disjoint Cycles

In this section we wish to explore the situation of when several cycles may be found in a graph. At this point in time we only care that these cycles are disjoint (both vertices and edges). We begin with a result due to Pósa [43]. Our concern is in finding a pair of vertex disjoint cycles of unspecified order. We define $s(n)$ to be the minimum number of edges so that every graph on *n* vertices contains two vertex disjoint cycles.

Theorem 5.8.1 For $n \geq 6$, $s(n) = 3n - 5$.

Proof. We first note that $3n - 6$ edges is not enough to guarantee the existence of two vertex disjoint cycles. To see this, we need only consider $K_{1, 1, 1, n-3}$ and note that any cycle in this graph must use at least two of the vertices in the partite sets of cardinality 1. Hence, two disjoint cycles cannot exist.

To establish that every $(n, 3n - 5)$ graph must contain two disjoint cycles, we proceed inductively on *n*. For $n = 6$, we note that $3n - 5 = 13$. This means that the graph in question is K_6 minus two edges. It is easy to check that no matter how these two edges are deleted, two vertex disjoint triangles remain.

Now, assume the result is true for all such graphs of order less than n and consider an $(n, 3n - 5)$ graph G. Since $\sum deg\ v_i = 6n - 10$, we see that G must have a vertex of degree at most 5. Call such a vertex x. We now consider cases for the possible values of $deg\ x$.

Suppose that $deg\ x = 5$ and let $N(x) = \{ v_1, v_2, \ldots, v_5 \}$. If $| E(<N[x]>) | \geq 13$, then as in the anchor step, G contains two disjoint triangles. Otherwise, some v_i, say v_1, has two nonadjacencies in $N(x)$. Without loss of generality, say v_1 is not adjacent to v_2 and v_3.

Now, consider the graph $H = G - x + \{ v_1 v_2, v_1 v_3 \}$. Note that H is an $(n - 1, 3n - 8)$ graph and so by the induction hypothesis, H must contain two disjoint cycles, say C_1 and C_2. If C_1 and C_2 do not use the edges we inserted to form H, then they must also exist in our original graph G. Thus, since at most one of these cycles can use the edges we inserted, we must have one cycle intact inside G. We now wish to show that the other cycle from H can be modified to form a second cycle in G.

If this cycle uses $v_1 v_2$ (or $v_1 v_3$) only, then replacing this edge with the vertex x and edges $v_1 x$, xv_2 produces the desired cycle in G. If both $v_1 v_2$ and $v_1 v_3$ are used, replace them with x and xv_2, xv_3 to form the desired cycle. In any case, two disjoint cycles in G are found.

If $deg\ x = 4$, a similar argument applies. If $deg\ x \leq 3$, then when x is removed from G, no other edges need be added to be able to apply the inductive hypothesis, and two disjoint cycles are found in $G - x$ immediately. Thus, in all cases, the result holds. □

This result is sharp since the graph $K_3 + nK_1$ has $p = n + 3$ vertices, $3p - 6$ edges and does not contain 2 disjoint cycles.

For $K_{1,3}$-free graphs, Matthews [38] proved that if $q \geq p + 6$, then the graph contains 2 disjoint cycles. We will prove a weaker (and much shorter) result.

Theorem 5.8.2 If G is a (p, q) $K_{1,3}$-free graph with $q \geq p + 6$, then G contains 2 disjoint cycles or $\Delta(G) < 6$.

Proof: Suppose that $\Delta(G) \geq 6$ and also that the result fails, hence any two cycles in G intersect. Since $deg\ u \geq 6$ for some vertex u, then by Theorem 1.6.2 and the fact that G is $K_{1,3}$-free, we see that $N(u)$ must contain a triangle, say T. But u and any three of the remaining vertices in $N(u) - V(T)$ cannot induce a $K_{1,3}$, thus a triangle disjoint from T exists. □

This result is also sharp since the graph K_5 with a path of length n attached to any one of its vertices has $p + 5$ edges, is $K_{1,3}$-free and does not contain 2 disjoint cycles.

For the case of finding k disjoint cycles, it was shown in [14] that for $k \geq 1$ and $p \geq 24k$, then every graph with $q \geq (2k - 1)p - 2k^2 + k$ contains either k disjoint cycles or $G = K_{2k-1} + (p - 2k + 1)K_1$. This result was improved for $K_{1,3}$-free graphs by Chen, Markus and Schelp [9].

Theorem 5.8.3 Let G be a $K_{1,3}$-free graph and let $k \geq 1$. If

$$q \geq p + (3k - 1)(3k - 4)/2 + 1$$

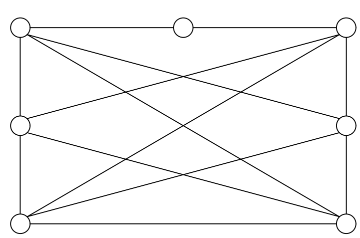
then G contains k disjoint cycles.

Finally, Corrádi and Hajnal [10] showed the following.

Theorem 5.8.4 If G is an (n,q) graph with $n \geq 3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles.

Exercises

1. Prove Corollary 5.1.2.
2. Prove Theorem 5.1.2.
3. Determine the complexity of Algorithm 5.1.1.
4. Use each of Algorithms 5.1.1, 5.1.2 and 5.1.3 to find an eulerian cycle in the graph below.



5. Determine an algorithm to accomplish the splitting away of two edges.
6. If H is the graph obtained from G by splitting away $e_1 = vw$ and $e_2 = vx$, prove that H is connected if, and only if, G is connected and $\{ e_1, e_2 \}$ does not form a cut set.

7. Prove that a nontrivial connected graph G is eulerian if, and only if, every edge of G lies on an odd number of cycles.
8. Prove that a nontrivial connected digraph D is eulerian if and only if E can be partitioned into subsets $E_1, E_2, \ldots, E_k$ such that the graph induced by E_i is a cycle for each i, $(1 \leq i \leq k)$.
9. Prove Observations 1 and 2.
10. Show that if $G = (V, E)$ is hamiltonian, then for every proper subset S of V, the number of components in $G - S$ is at most $|S|$.
11. Show that if G is not 2-connected, then G is not hamiltonian.
12. Characterize when the graph $K_{p_1, p_2, \ldots, p_n}$ is hamiltonian.
13. Prove or disprove: If G and H are hamiltonian, then $G \times H$ is hamiltonian.
14. Prove or disprove: If G and H are hamiltonian, then $G[H]$ is hamiltonian.
15. Let the n-cube be the graph $Q_n = K_2 \times Q_{n-1}$ (where $Q_1 = K_2$). Prove that if $n \geq 2$, Q_n is hamiltonian.
16. Let G be a graph with $\delta(G) \geq 2$. Show that G contains a cycle of length at least $\delta(G) + 1$.
17. Suppose that G is a (p, q) graph with $p \geq 3$. Show that if $q \geq \dfrac{p^2 - 3p + 6}{2}$, then G is hamiltonian.
18. Show that if G is a (p, q) graph with $q \geq \binom{p-1}{2} + 3$, then G is hamiltonian connected.
19. Show that if G is hamiltonian connected, then G is 3-connected.
20. Show that if a (p, q) graph G is hamiltonian connected and if $p \geq 4$, then $q \geq \left\lfloor \dfrac{3p + 1}{2} \right\rfloor$.
21. Give an example of a graph that is pancyclic but not panconnected.
22. Find an example of a graph that is hamiltonian connected but not panconnected.
23. Find an example of a graph that is pancyclic but not vertex pancyclic.

24. Can we remove the restriction that D be strongly connected from Meyniel's theorem?

25. Show that every complete graph with directed edges is traceable.

26. Show that K_n with strong directed edges is vertex pancyclic.

27. Give an example of a hamiltonian connected digraph that satisfies the conditions of Theorem 5.4.2 but does not have $od\ v \geq \frac{p+1}{2}$ and $id\ v \geq \frac{p+1}{2}$ for every vertex v.

28. Show that the Petersen graph is homogeneously traceable nonhamiltonian and also hypohamiltonian.

29. Show that homogeneously traceable nonhamiltonian graphs exist for all orders $p \geq 9$.

30. Show that if $G = (V, E)$ is a homogeneously traceable nonhamiltonian graph and $x \in V$, then x is adjacent to at most one vertex of degree 2.

31. Show that if $C_{p-1}(G) = K_p$, then G is traceable.

32. Prove Corollary 5.4.2.

33. Prove that the graph G^2 of Figure 5.5.1 is not hamiltonian.

34. Show (without using Fleischner's theorem) that if G is 2-connected, then G^3 is hamiltonian. (Hint: Consider spanning trees).

35. Use Fleischner's theorem to show that if G is 2-connected, then G^2 is hamiltonian connected. (Hint: Consider five copies of G along with two additional vertices x and y joined to an arbitrary pair of vertices u and v in each copy of G).

36. Prove Theorem 5.5.2.

37. Prove that if G is a graph of order $p \geq 3$ such that the vertices of G can be labeled $v_1, v_2, \ldots, v_p$ so that

$$\begin{array}{c} j<k,\ j+k \geq p,\ v_j v_k \notin E(G) \\ deg\ v_j \leq j,\ deg\ v_k \leq k-1 \end{array} \Rightarrow deg\ v_j + deg\ v_k \geq p.$$

then G is hamiltonian.

38. Let G be a graph of order $p \geq 3$, the degrees d_i of whose vertices satisfy $d_1 \leq d_2 \leq, \ldots, \leq d_p$. If

$$d_j \leq j < \frac{p}{2} \Rightarrow d_{p-j} \geq p - j,$$

then G is hamiltonian.

39. Prove that if G is a graph of order $p \geq 3$ such that for every integer j with $1 \leq j < \frac{p}{2}$, the number of vertices of degree not exceeding j is less than j, then G is hamiltonian.

40. Prove that if G has order $p \geq 3$ and if $k(G) \geq \beta(G)$ = the maximum number of mutually nonadjacent vertices, then G is hamiltonian.

41. (Ghouila-Houri [25]) Let D be a strongly connected digraph such that $deg\ x \geq p$ for every vertex x of D. Prove D is hamiltonian.

42. (Woodall [48]) Let D be a digraph of order $p \geq 3$ such that whenever x and y are distinct vertices and $x \rightarrow y$ is not an arc of D, then $od\ x + id\ y \geq p$. Prove D is hamiltonian.

43. (Meyniel [39]) Let D be a strongly connected digraph of order $p \geq 3$ such that for every distinct pair of nonadjacent vertices x and y, $deg\ x + deg\ y \geq 2p - 1$. Prove D is hamiltonian.

44. (Overbeck-Larisch [42]) Let D be a digraph of order $p \geq 2$ such that for every pair of distinct vertices x and y such that $x \rightarrow y$ is not an arc of D, $od\ x + id\ y \geq p + 1$. Prove D is hamiltonian connected.

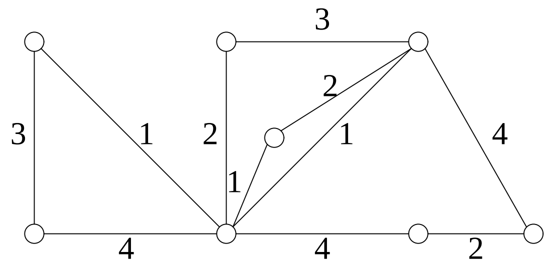
45. Determine all pairs of graphs (R, S) that when forbidden, imply a 2-connected graph is pancyclic.

46. Determine all pairs of graphs (R, S) that when forbidden, imply a 2-connected graph is panconnected.

47. Determine all pairs of graphs (R, S) that when forbidden, imply a connected graph is traceable.

48. Show that the only single graph that when forbidden, implies a 2-connected graph is hamiltonian, is P_3.

49. Determine the minimum salesman's walk in the following graph.



50. Show that the greedy approach to the traveling salesman problem can be arbitrarily bad.

51. Prove that the graph of Figure 5.7.1 is the unique $(5, 3)$-cage and is isomorphic to the Petersen graph.

52. Find the $(6, 3)$-cage and show that it is unique.

53. Prove Theorem 5.7.4(1).

54. Prove that if G is a $K_{1,3}$-free graph that does not contain k disjoint cycles, then $\Delta(G) \leq 3k - 1$.

55. Prove that if G is a $K_{1,3}$-free graph with $\Delta(G) \leq 5$ that G contains two disjoint cycles.

56. Prove that if a (p, q)-graph G is $K_{1,3}$-free, $k \geq 1$ and $q \geq p + (3k - 1)(3k - 4)/2 + 1$, then G contains k disjoint cycles.

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