THE DISTRIBUTION OF NEGATIVE EIGENVALUES OF SCHRÖDINGER OPERATORS ON ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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ABSTRACT. We study the asymptotic behavior of the counting function of negative eigenvalues of Schrödinger operators with real valued potentials on asymptotically hyperbolic manifolds. We establish conditions on the potential that determine if there are finitely or infinitely many negative eigenvalues. In the latter case, they may only accumulate at zero and we obtain the asymptotic behavior of the counting function of eigenvalues in an interval $(-\infty, -E)$ as $E \to 0$.

1. INTRODUCTION

We are concerned with the following type of problem: Let (X, g) be a C^∞ non-compact complete C^∞ Riemannian manifold and let Δ_g be its (positive) Laplacian. Suppose that V is a real valued potential such that $V < 0$ near infinity, $H = \Delta_g + V$ is self-adjoint, its point spectrum $\sigma_p(H) \subset (-E_0, 0)$ and the eigenvalues only accumulate at zero. The problem is to find conditions on V which determine whether the point spectrum is finite or infinite and if it is infinite, determine the asymptotic behavior of the number of eigenvalues (counted with multiplicity) in an interval $(-\infty, -E)$ as $E \to 0$.

In the Euclidean case, it has been shown, see for example [24], that if V is bounded and if that near infinity $V(z) \leq -C|z|^{-2+\delta}$, $\delta > 0$, then H has infinitely many eigenvalues, while if $V \geq -C|z|^{-2-\delta}$, $\delta > 0$ there are finitely many eigenvalues. The threshold decay of $V(z)$ for H to have finitely or infinitely many eigenvalues is therefore $V(z) \sim F(\omega)|r|^{-2}$, $r = |z|$, $z = r\omega$, $\omega \in \mathbb{S}^{n-1}$. Moreover, an application of Hardy's inequality shows that if $V(z) \sim -cr^2$, there are finitely many eigenvalues when $c < \frac{(n-1)^2}{4}$ and infinitely many if $c > \frac{(n-1)^2}{4}$. Precise results on the asymptotics of the counting function of eigenvalues as $E \to 0$, in the case where $V(z) = r^{-2}(F(\omega) + \mathcal{E}(r, \omega))$ with $\mathcal{E}(r, \omega) = o((\log r)^{-1-\varepsilon})$ as $r \to \infty$ were obtained by Kirsch and Simon [14] and Hassell and Marshal [10]. We are not aware of similar results for asymptotically Euclidean manifolds.

While this problem has been well studied in the Euclidean space, it seems that it has not been studied as much in hyperbolic space. We will work in the class of asymptotically hyperbolic manifolds in the sense of [17], for which the hyperbolic space serves as a model.

The Poincaré model of hyperbolic space $\mathbb{H}^{n+1}$ is given by the Euclidean ball of radius one

$$\mathbb{B}^{n+1} = \{z \in \mathbb{R}^{n+1} : |z| < 1\} \text{ equipped with the metric } g_0(z) = \frac{4dz^2}{(1 - |z|^2)^2}, \tag{1.1}$$

which is the interior of a C^∞ manifold equipped with a metric which is singular at its boundary.

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The function $\rho(z) = 1 - |z|^2 \in C^\infty(\overline{\mathbb{B}^{n+1}})$, $\rho^{-1}(0) = \mathbb{S}^n = \partial\mathbb{B}^{n+1}$ and $d\rho(z) \neq 0$ if $|z| = 1$, and so it is called a defining function of $\partial\mathbb{B}^{n+1}$, and moreover $\rho^2 g_0 = 4dz^2$ is a C^∞ Riemannian metric on the closure $\overline{\mathbb{B}^{n+1}}$.

Throughout this paper, $\mathring{X}$ will denote the interior of a C^∞ compact manifold X with boundary ∂X of dimension $n+1$. We say that $\rho \in C^\infty(X)$ is a defining function of ∂X, or a boundary defining function, if $\rho > 0$ in $\mathring{X}$, $\rho = 0$ at ∂X and $d\rho \neq 0$ at ∂X. We assume $\mathring{X}$ is equipped with a C^∞ Riemannian metric g such that $G = \rho^2 g$ is non-degenerate at ∂X and so $(\bar{X}, G)$ is a C^∞ compact Riemannian manifold with boundary. According to [15], the sectional curvature of $(\mathring{X}, g)$ converges to $-|d\rho|_G$ along any curve that goes towards ∂X. The manifold $(\mathring{X}, g)$ is called an asymptotically hyperbolic manifold, or AHM, if and $|d\rho|_G = 1$ at ∂X. One might relax this assumption if X has more than one boundary component and $\partial X = Y_1 \sqcup Y_2 \ldots \sqcup Y_N$ and $|d\rho|_G = \kappa_j$ at Y_j, κ_j is a constant.

The hyperbolic space serves as the model for this class of manifolds, and its quotients by certain discrete groups of fractional linear transformations having a geometrically finite fundamental domain without cusps at infinity are also examples of such manifolds, see [1,17,21,22]. In fact our results apply for manifolds with more than one boundary component and with different (constant) asymptotic curvatures at each end. One important example from mathematical physics where this occurs is the stationary model of the de Sitter- Schwarzschild model of black holes discussed in [25]. In this case the manifold is $\mathring{X} = (a, b) \times \mathbb{S}^n$, and the asymptotic curvatures are different on both ends.

Let Δ_g denote the (positive) Laplace operator on an AHM $(\mathring{X}, g)$. We know from [15, 16], see also [3] that the spectrum of Δ_g, denoted by $\sigma(\Delta_g)$, satisfies $\sigma(\Delta_g) = \sigma_{pp}(\Delta_g) \cup \sigma_{ac}(\Delta_g)$, where $\sigma_{pp}(\Delta_g)$ consists of a finite number of eigenvalues in $(0, \frac{n^2}{4})$ and $\sigma_{ac} = [\frac{n^2}{4}, \infty)$ is the absolutely continuous spectrum. We shall work with $\Delta_g - \frac{n^2}{4}$ which has continuous spectrum $[0, \infty)$. Let $V \in L^\infty(X)$ be real valued and such that $V(z) < 0$ near ∂X and $V(z) \to 0$ as $z \to \partial X$. We shall denote

$$H_0 = \Delta_g - \frac{n^2}{4} \text{ and } H = \Delta_g - \frac{n^2}{4} + V. \tag{1.2}$$

We will show that under the assumptions on the rate of decay of the potential, the point spectrum of H satisfies $\sigma_{\mathrm{pp}}(H) \subset [-E_0, 0)$, consists of eigenvalues of finite multiplicity and can only accumulate at zero and its essential spectrum $\sigma_{ess}(H) = [0, \infty)$, with no embedded eigenvalues, and the bottom of the spectrum is not an eigenvalue. We want to count negative eigenvalues of H, and so for $E \geq 0$, we define

$$N_E(H) = \#\{\mu_j \in (-\infty, -E), E \geq 0 : \mu_j \text{ is an eigenvalue of } H, \text{ counted with multiplicity}\} \tag{1.3}$$

If $(\mathring{X}, g)$ is an AHM and $G = \rho^2 g$, then the metric $G|_{\partial X} = h_0$, depends on the choice of ρ. In fact, given any two boundary defining functions ρ and $\tilde{\rho}$, we must have $\rho = a(z)\tilde{\rho}$, with $a > 0$. If $\tilde{G} = \tilde{\rho}^2 g$, then $G = a^2 \tilde{G}$, and in particular $G|_{\partial X} = (a^2 \tilde{G})|_{\partial X}$, and hence $\rho^2 g$ determines a conformal class of metrics at ∂X. As shown in [9,13], given a representative h_0 of the class $[\rho^2 g|_{\partial X}]$ there exists a unique boundary defining function x, a neighborhood U of ∂X and a map $\Psi : [0, \varepsilon) \times \partial X \longrightarrow U$ such that

$$\Psi^* g = \frac{dx^2}{x^2} + \frac{h(x)}{x^2}, \quad h(0) = h_0, \tag{1.4}$$

where $h(x)$ is a C^∞ one-parameter family of Riemannian metrics on ∂X.

For example, in the case of the hyperbolic space (1.1), the geodesic distance with respect to the origin is given by

$$r = \log\left(\frac{1+|z|}{1-|z|}\right),$$

the metric g_0 is given by

$$g_0 = dr^2 + (\sinh r)^2 d\theta^2,$$

where $d\theta^2$ is the standard metric on the sphere. If we set $x = e^{-r}$, then

$$g_0 = \frac{dx^2}{x^2} + \frac{(1-x^2)^2}{4}\frac{d\theta^2}{x^2},$$

In coordinates for which (1.4) is valid, the Laplacian with respect to g is given by

$$\Delta_g = -(x\partial_x)^2 - nx\partial_x - x^2 A(x,y)\partial_x + x^2\Delta_{h(x)}, \tag{1.5}$$

where $\Delta_{h(x)}$ is the Laplacian with respect to the metric $h(x)$ on ∂X, and $A = \frac{1}{2}\partial_x \log|h|$, where $|h|$ is the volume element of the metric h. It is convenient to set $x = e^{-\rho}$, and so

$$\Delta_g = -\partial_\rho^2 - n\partial_\rho - e^{-\rho}\mathcal{A}(\rho,y)\partial_\rho + e^{-2\rho}\Delta_{\widetilde{h}(\rho)}, \tag{1.6}$$

where $\mathcal{A}(\rho,y) = A(e^{-\rho},y)$ and $\widetilde{h}(\rho) = h(e^{-\rho})$.

From now on, $(\mathring{X}, g)$ will denote a $n+1$ dimensional asymptotically hyperbolic manifold and x is a boundary defining function such that (1.4) holds and $x = e^{-\rho}$. We assume that $V \in L^\infty(X)$ *is real valued* and we let $H = \Delta_g + V - \frac{n^2}{4}$ and $N_E(H)$ be defined as above.

Theorem 1.1. *Suppose that in some interval $\rho \in (\rho_0, \infty)$, there exists a constant $c > 0$ such that*

$$V(e^{-\rho}, y) = -c\rho^{-2+\delta} + O(\rho^{-2+\delta}(\log\rho)^{-\varepsilon}), \quad \varepsilon > 0 \text{ and } \delta < 2, \text{ as } \rho \to \infty. \tag{1.7}$$

If $\delta < 0$, then $N_0(H) < \infty$, but if $\delta \in (0,2)$, then $N_0(H) = \infty$ and

$$\log\log N_E(H) = \frac{1}{2-\delta}\log E^{-1} + O(1), \text{ as } E \to 0. \tag{1.8}$$

One could choose the error to be $O(\rho^{-2+\delta}f(\rho))$ such that Theorem A.2 can be applied; we pick $f(\rho) = (\log\rho)^{-\varepsilon}$ for convenience. Notice that if $\widetilde{x} = e^{\varphi(x,y)}x$ is another boundary defining function, then

$$\widetilde{\rho} = -\log\widetilde{x} = -\varphi - \log x = \rho + O(1), \text{ as } \rho \to \infty,$$

so (1.7) does not depend on the choice of x.

In the threshold case $\delta = 0$, case we have the following

Theorem 1.2. *Suppose that in some interval $\rho \in (\rho_0, \infty)$,*

$$V(e^{-\rho}, y) = -c\rho^{-2} + o(\rho^{-2}(\log\rho)^{-\varepsilon}), \ c > 0, \quad \varepsilon > 0, \text{ as } \rho \to \infty. \tag{1.9}$$

If $c < \frac{1}{4}$, then $N_0(H) < \infty$, but if $c > \frac{1}{4}$, then $N_0(H) = \infty$ and

$$\log\log N_E(H) = -\frac{1}{2}\log E + O(1) \text{ as } E \to 0. \tag{1.10}$$

Our proofs in fact give somewhat more precise upper and lower bounds for $N_E(H)$, and (1.8) and (1.10) are used to unify these bounds and provide the asymptotic behavior of iterated logarithms of $N_E(H)$.

The methods we use do not allow us to treat the case where c is a function of y. However, we can use these results to prove

Corollary 1.3. *Suppose that in some interval $\rho \in (\rho_0, \infty)$,*

$$-c_1\rho^{-2+\delta} \leq V(e^{-\rho}, y) \leq -c_2\rho^{-2+\delta}, \tag{1.11}$$

then we can say that

1. *If $\delta < 0$, then $N_0(H) < \infty$.*
2. *If $\delta \in (0,2)$, then $N_0(H) = \infty$ and (1.8) holds.*
3. *If $\delta = 0$ and $c_1 < \frac{1}{4}$, then $N_0(H) < \infty$.*
4. *If $\delta = 0$ and $c_2 > 1/4$, then $N_0(H) = \infty$ and (1.10) holds.*

We can say more in the threshold case $c = \frac{1}{4}$. For ρ large enough, we define

$$\log_{(j)} \rho = \log\log \dots \log \rho, \text{ j times.} \tag{1.12}$$

Theorem 1.4. *Suppose that in some interval $\rho \in (\rho_0, \infty)$, there exists a constant $c_1 > 0$ such that*

$$V(e^{-\rho}, y) = -\frac{1}{4}\rho^{-2} - c_1\rho^{-2}(\log\rho)^{-2} + O\left(\rho^{-2}(\log\rho)^{-2}(\log\rho)^{-\varepsilon}\right), \; \varepsilon > 0. \tag{1.13}$$

If $c_1 < \frac{1}{4}$, then $N_0(H) < \infty$ and if $c_1 > \frac{1}{4}$, $N_0(H) = \infty$ and

$$\log_{(3)} N_E(H) = \log_{(2)}\left(E^{-1}\right) + O(1) \text{ as } E \to 0. \tag{1.14}$$

In fact this process keeps going indefinitely and the result holds if for some ρ_0 large, and $\rho \in [\rho_0, \infty)$, the potential has an expansion of the form

$$\begin{gathered} V(e^{-\rho}, y) = V_0(\rho) + O\left(\mathcal{G}_N(\rho)(\log\rho)^{-\varepsilon}\right), \;\; \varepsilon > 0 \text{ where} \\ V_0(\rho) = -\frac{1}{4}\rho^{-2} - \frac{1}{4}\sum_{j=1}^{N-1}\mathcal{G}_j(\rho) + c_N\mathcal{G}_N(\rho), \\ \mathcal{G}_{(j)}(\rho) = \rho^{-2}(\log\rho)^{-2}(\log\log\rho)^{-2}\dots(\log_{(j)}\rho)^{-2}, \end{gathered} \tag{1.15}$$

where c_N is a constant. The existence of infinitely many eigenvalues depend on whether $c_N < \frac{1}{4}$, or $c_N > \frac{1}{4}$. If $c_N < \frac{1}{4}$ there are only finitely many eigenvalues, but if $c_N > \frac{1}{4}$,

$$\log_{(N+2)} N_E(H) = \log_{(N+1)}(E^{-1}) + O(1) \text{ as } E \to 0. \tag{1.16}$$

Notice that the limit of $V_N(\rho)$ as $N \to \infty$ cannot be well defined for all $\rho \in [\rho_0, \infty)$ for some ρ_0 because the denominators will be equal to zero at points of the form $\rho = e^{e^{e^{\cdots}}}$.

As in the case of Theorems 1.1 and 1.2, our proofs actually give better upper and lower bounds for $N_E(H)$ and this formulation is used to unify these bounds.

One should mention the work of Mazzeo and McOwen [18]. While the problems they study are somewhat the opposite of the ones we study here, there are some similarities.

1.1. **The Strategy of the Proofs.**

The methods used in the proof of Theorems 1.1, 1.2 and 1.4 are Dirichlet-Neumann bracketing and the Sturm oscillation theorem, which are standard for this type problems.

For $\varepsilon_0 > 0$, let

$$X_\infty = \{z \in X : x(z) \leq \varepsilon_0\}, \quad X_0 = \{z \in X : x(z) \geq \varepsilon_0\}. \tag{1.17}$$

So X_0 is compact and X_∞ is a collar neighborhood of ∂X. Let $\Upsilon_0^\bullet$, and $\Upsilon_\infty^\bullet$, be the restrictions of the operator H to X_0 and X_∞ with Dirichlet ($\bullet = D$) and Neumann ($\bullet = N$) boundary conditions. Since X_0 is a C^∞ Riemannian manifold with boundary, it is well known that

$$\begin{aligned} \sigma(\Upsilon_0^D) &= \{\lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \ldots\}, \quad \lambda_j \in \mathbb{R}, \quad \lambda_j \to \infty, \\ \sigma(\Upsilon_0^N) &= \{\mu_1 < \mu_2 \leq \mu_3 \leq \mu_4 \ldots\}, \quad \mu_j \in \mathbb{R}, \quad \mu_j \to \infty. \end{aligned} \tag{1.18}$$

It is known that $\sigma_{ess}(\Upsilon_\infty^\bullet) = [0, \infty)$, $\bullet = D, N$, and no embedded eigenvalues, and the point spectra

$$\begin{aligned} \sigma_{pp}(\Upsilon_\infty^D) &= \{\lambda_1 < \lambda_2 \leq \lambda_3 \ldots\} \;\; \lambda_j < 0, \text{ is finite or } \lambda_j \to 0, \\ \sigma_{pp}(\Upsilon_\infty^N) &= \{\mu_1 < \mu_2 \leq \mu_3 \ldots\} \;\; \mu_j < 0, \text{ is finite or } \mu_j \to 0. \end{aligned} \tag{1.19}$$

Following Chapter XIII of [24], we will show that

$$\Upsilon_0^N \oplus \Upsilon_\infty^N \leq H \leq \Upsilon_0^D \oplus \Upsilon_\infty^D, \tag{1.20}$$

and it follows that for any $E < 0$,

$$N_E(\Upsilon_0^N) + N_E(\Upsilon_\infty^N) \leq N_E(H) \leq N_E(\Upsilon_0^D) + N_E(\Upsilon_\infty^D). \tag{1.21}$$

But it follows from (1.18) that there exists $N^\# > 0$ such that for any $E < 0$, $N_E(Q_0^N) < N^\#$ and $N_E(Q_0^D) < N^\#$. We will show that if V satisfies the hypotheses of either one of the Theorems 1.1, 1.2 and 1.4, then for $E < 0$, both $N_E(\Upsilon_\infty^\bullet)$, $\bullet = N, D$, have either finitely or infinitely many eigenvalues. In case both have infinitely many eigenvalues, the corresponding counting function of their eigenvalues have the same asymptotic behavior as $E \searrow 0$, and therefore it gives the asymptotic behavior of $N_E(H)$.

2. The Spectrum of H

For the lack of suitable references, we shall briefly discuss some properties of the spectrum of H. First we recall some results about of the spectrum of H_0 from from [16, 17]. Let x be a boundary defining function for which (1.4) holds in a collar neighborhood of ∂X. In these coordinates, the Laplacian Δ_g is given by (1.6), so Δ_g is a zero differential operator in the sense of [17], and we define the zero-Sobolev spaces of order k as in [17]: Let $\mathcal{V}(\partial X)$ denote the Lie algebra of C^∞ vector fields on X which are equal to zero at ∂X. In coordinates (x, y) these vector fields are spanned by $\{x\partial_x, x\partial_{y_j},\ 1 \leq j \leq n\}$ over the C^∞ functions. Let

$$\mathcal{H}_0^k(X) = \{u \in L^2(X) : \;\; W_1 W_2 \ldots W_m u \in L^2(X), \;\; W_j \in \mathcal{V}(\partial X), \;\; m \leq k\}. \tag{2.1}$$

We know from the work of Mazzeo and Melrose [17] that Δ_g with domain $\mathcal{H}_0^2(X) \subset L^2(X)$ is a self-adjoint operator, we also know from [17], [15] and [3] that its spectrum consists of an absolutely continuous part $\sigma_{ac}(\Delta_g) = [\frac{n^2}{4}, \infty)$ and finitely many eigenvalues in the point spectrum $\sigma_{pp}(\Delta_g) \subset (0, \frac{n^2}{4})$. As above, we set $H_0 = \Delta_g - \frac{n^2}{4}$, and so the resolvent

$$\begin{gathered} R_{H_0}(\lambda) = (H_0 - \lambda)^{-1} : L^2(X) \longmapsto \mathcal{H}_0^2(X), \\ \text{provided } \lambda \in \mathbb{C} \setminus ([0, \infty) \cup \{\lambda_1, \lambda_2, \ldots, \lambda_N\}), \end{gathered}$$

where $\lambda_j \in (-\frac{n^2}{4}, 0)$ is an eigenvalue of finite multiplicity of H_0. By definition, the resolvent set of H_0 is

$$\rho(H_0) = \; \mathbb{C} \setminus ([0, \infty) \cup \{\lambda_1, \ldots \lambda_N\}), \;\; \lambda_j \text{ is an eigenvalue of } H_0. \tag{2.2}$$

To analyze the spectrum of H, we begin by observing that for $\lambda \in \rho(H_0)$,

$$(\Delta_g + V - \frac{n^2}{4} - \lambda)R_{H_0}(\lambda) = \mathrm{I} + VR_{H_0}(\lambda), \tag{2.3}$$

Since

$$R_{H_0}(\lambda) : L^2(X) \longrightarrow \mathcal{H}_0^2(X) \text{ is a bounded operator for } \lambda \in \rho(H_0),$$

then if $\chi_j \in C_0^\infty(\mathring{X})$, $\chi_j(z) = 1$ in the region $x(z) > \frac{1}{j}$ and $\chi_j(z) = 0$ if $x(z) < \frac{1}{j+1}$, it follows that

$$\chi_j(z)V(z)R_{H_0}(\lambda) : L^2(X) \longrightarrow H_c^2(\mathring{X}) \hookrightarrow L_c^2(\mathring{X}),$$

where the subindex c indicates compact support. Notice that since supports are compact we can use either $\mathcal{H}_0^2(X)$ or the standard Sobolev space $H^2(X)$. It follows from Rellich's embedding Theorem that for fixed j,

$$\chi_j(z)V(z)R_{H_0}(\lambda) : L^2(X) \longrightarrow L_c^2(\mathring{X})$$

is a compact operator. Since $V \in L^\infty(X)$ and $V(z) \to 0$ as $z \to \partial X$, it follows that

$$||\chi_j(z)V(z)R_{H_0}(\lambda) - V(z)R_{H_0}(\lambda)||_{\mathcal{L}(L^2(X))} \to 0 \text{ as } j \to \infty$$

in the operator norm, and so we conclude that $VR_{H_0}(\lambda) : L^2(X) \longrightarrow L^2(X)$ is a compact operator, provided $\lambda \in \rho(H_0)$. But we also know that for $\operatorname{Im}\lambda << 0$, the operator norm of $R_{H_0}(\lambda)$ is less than or equal to $\frac{1}{\operatorname{Im}(\lambda)}$, see for example Theorem VI.8 of [23], and therefore $(\mathrm{I} + VR_{H_0}(\lambda))^{-1}$ is bounded for $\operatorname{Im}\lambda << 0$, and $|\operatorname{Re}\lambda| > 1$. Then Fredholm Theorem, see for example Theorem VI.14 of [23], guarantees that with the exception of a countable set of points, which are poles of $R_H(\lambda)$,

$$R_H(\lambda) = R_{H_0}(\lambda)(\mathrm{I} + VR_{H_0}(\lambda))^{-1} \text{ for } \lambda \in \rho(H_0).$$

Moreover the poles of $R_H(\lambda)$ in $\rho(H_0)$ consist of a countable set $\{\mu_j, \;\; j \in \mathbb{N}\} \subset (-\infty, 0)$ such that μ_j are eigenvalues of H with finite multiplicity. This set is either finite, or infinite. If there are infinitely may eigenvalues, they accumulate only at zero.

Finally, since $VR_{H_0}(\lambda)$ is compact for $\lambda \in \rho(H_0)$, it follows that the operator

$$\begin{aligned} V : L^2(X) &\longmapsto L^2(X) \\ f &\longmapsto V(z)f \end{aligned}$$

is relatively compact with respect to H_0 and it follows from Weyl's Theorem, see Theorem 14.6 of [11] that $\sigma_{ess}(H) = \sigma(H) \setminus \sigma_{pp}(H) = \sigma_{ess}(H_0) = [0, \infty)$.

Therefore we have

Theorem 2.1. *Let $(\mathring{X}, g)$ be an AHM, let $V \in L^\infty(X)$ be real valued and suppose that $V(z) \to 0$ as $z \to \partial X$. Then $\sigma_{ess}(H) = \sigma(H) \setminus \sigma_{pp}(H) = [0, \infty)$. Moreover, the resolvent*

$$R_H(\lambda) = (H - \lambda)^{-1} : L^2(X) \longrightarrow \mathcal{H}_0^2(X) \text{ is bounded for } \lambda \in \mathbb{C} \setminus ([0, \infty) \cup \mathcal{D}),$$

where $\mathcal{D} = \{\mu_1, \mu_2, \dots\} \subset (-\infty, 0)$, with $\mu_{j+1} \geq \mu_j$, is a bounded discrete set (in $(-\infty, 0)$) which consists of eigenvalues of H of finite multiplicity.

We also have the following

Theorem 2.2. *Let $(\mathring{X}, g)$ be an AHM, let x be a boundary defining function such that (1.4) holds. Let $X_\infty = \{z \in X : x(z) \leq x_0\}$ and let $\Upsilon_\infty^\bullet$, $\bullet = N, D$ denote the operator H with Dirichlet or Neumann boundary conditions in X_∞. If $V(z) \to 0$ as $z \to \partial X$, then $\sigma_{ess}(\Upsilon_\infty^\bullet) = [0, \infty)$. Moreover, there are no embedded eigenvalues.*

The fact that $\sigma_{ess}(\Upsilon^\bullet_\infty) = [0,\infty)$ is a consequence of Theorem 2.1 and Proposition 2.1 of [5], see also Theorem 9.43 of [2]. The proofs of these results are actually for the Dirichlet boundary conditions, but they work for Neumann conditions as well. The fact that there are no embedded eigenvalues is due to Mazzeo [16], see also [2,3]. The point is to show that if there were eigenfunctions in L^2, they would decay exponentially and a Carleman estimate shows that they are actually equal to zero. The argument takes place in a neighborhood of ∂X and also works for $\Upsilon^\bullet_\infty$.

3. Dirichlet-Neumann Bracketing

One can view the operator H as the unique self-adjoint operator on $L^2(X)$ whose quadratic form is the closure of

(3.1)
$$Q(\varphi,\psi) = \langle \nabla_g\phi, \nabla_g\psi\rangle_{L^2_g(X)} + \langle (V-\frac{n^2}{4})\phi,\psi\rangle_{L^2_g(X)} = \int_X g^{ij}\partial_i\phi\partial_j\bar{\psi} d\operatorname{vol}_g + \int_X (V-\frac{n^2}{4})\phi\bar{\psi} d\operatorname{vol}_g,$$
$$\text{with } \phi,\psi\in C_0^\infty(\mathring{X}).$$

The domain of the quadratic form Q is $\mathcal{H}^1_0(X)\times\mathcal{H}^1_0(X)$, defined in (2.1).

Let x be a boundary defining function such that (1.4) holds and for $\varepsilon>0$ let X_0 and X_∞ be as defined in (1.17). We consider the quadratic forms to be the closure of

$$\begin{aligned} &Q^D(X_0)(\varphi,\psi) = Q(\varphi,\psi) \text{ with } \varphi,\psi\in C_0^\infty(\mathring{X}_0),\\ &Q^N(X_0)(\varphi,\psi) = Q(\varphi,\psi) \text{ with } \varphi,\psi\in C^\infty(X_0),\quad \partial_x\varphi|_{\{x=x_0\}} = \partial_x\psi|_{\{x=x_0\}} = 0. \end{aligned} \tag{3.2}$$

It turns out that the domains of these quadratic forms are

$$\begin{aligned} &\mathcal{D}(Q^D(X_0)) = H^1_0(X_0)\times H^1_0(X_0),\quad H^1_0(X_0) = \overline{C_0^\infty(\mathring{X}_0)} \text{ with the } H^1_{\text{loc}}(\mathring{X}) \text{ norm},\\ &\mathcal{D}(Q^N(X_0)) = \bar{H}^1(X_0)\times\bar{H}^1(X_0),\quad \bar{H}^1(X_0) = \{\varphi\in L^2(X_0) : \exists\, f\in H^1_{\text{loc}}(\mathring{X}),\ f=\varphi \text{ in } X_0\} \end{aligned}$$

The self-adjoint operator operators corresponding to $Q^D(X_0)$ and $Q^N(X_0)$ are defined to be the operator H respectively with Dirichlet and Neumann boundary conditions, which we denote by Υ^D_0 and Υ^N_0 respectively.

Similarly, we define the quadratic forms

$$\begin{aligned} &Q^D(X_\infty) = Q(\varphi,\psi),\quad \varphi,\psi\in C_0^\infty(\mathring{X}_\infty),\\ &Q^N(X_\infty) = Q(\varphi,\psi),\quad \varphi,\psi\in C^\infty(X_\infty)\cap\mathcal{H}^2_0(X_\infty)\ \ \partial_x\varphi|_{\{x=x_0\}} = \partial_x\psi|_{\{x=x_0\}} = 0. \end{aligned} \tag{3.3}$$

The domains of their closure are

$$\begin{aligned} &\mathcal{D}(Q^D(X_\infty)) = \mathcal{W}^1_0(X_\infty)\times\mathcal{W}^1_0(X_\infty),\ \mathcal{W}^1_0(X_\infty) = \overline{C_0^\infty(\mathring{X}_\infty)} \text{ with the } \mathcal{H}^1_0(X) \text{ norm}\\ &\mathcal{D}(Q^N(X_\infty)) = \bar{\mathcal{H}}^1_0(X_\infty)\times\bar{\mathcal{H}}^1_0(X_\infty),\ \bar{\mathcal{H}}^1_0(X_\infty) = \{\varphi\in L^2(X_\infty) : \exists f\in\mathcal{H}^1_0(X); f=\varphi \text{ in } X_\infty\}. \end{aligned}$$

The corresponding self-adjoint operators are defined to be Υ^D_∞ and Υ^N_∞ which are the operator H with Dirichlet or Neumann boundray conditions on X_∞.

We follow section XIII.15 of [24] and define the direct sum of self-adjoint operators: If A_j, $j=1,2$, are self adjoint operators acting on Hilbert spaces $\mathcal{L}_j$, $j=1,2$, with domains $\mathcal{D}(A_j)$, $j=1,2$, let

$$\mathcal{L} = \mathcal{L}_1\oplus\mathcal{L}_2 \text{ and } A_1\oplus A_2(\phi_1,\phi_2) = (A_1\phi_1, A_2\phi_2), \phi_j\in\mathcal{D}(A_j),\quad j=1,2.$$

It is proved in section XII.15 of [24] that

1. $A_1\oplus A_2$ is self adjoint.

2. The associated quadratic forms satisfy $Q(A_1 \oplus A_2) = Q(A_1) \oplus Q(A_2)$.
3. If $N(\lambda, A) = \dim P_{(-\infty,\lambda)}(A)$, then

$$N(\lambda, A_1 \oplus A_2) = N(\lambda, A_1) + N(\lambda, A_2). \tag{3.4}$$

In our case, we have four natural operators $\Upsilon^\bullet(X_0)$ and $\Upsilon^\bullet(X_\infty)$, $\bullet = D, N$. Notice that

$$\begin{gathered} H_0^1(X_0) \oplus \mathcal{W}_0^1(X_\infty) \subset \mathcal{H}_0^1(X), \text{ and that for } \quad \varphi \in C_0^\infty(\mathring{X}_0), \quad \psi \in C_0^\infty(\mathring{X}_\infty), \\ Q^D(X_0)(\varphi,\varphi) + Q^D(X_\infty)(\psi,\psi) = Q(H)(\varphi,\varphi) + Q(H)(\psi,\psi). \end{gathered}$$

On the other hand, if $\varphi \in \mathcal{H}_0^1(X)$,

$$\varphi|_{X_0} \in \bar{\mathcal{H}}_0^1(X_0) \text{ and } \varphi|_{X_\infty} \in \bar{\mathcal{H}}_0^1(X_\infty),$$

this means that

$$\begin{gathered} \mathcal{H}_0^1(X_0) \subset \bar{\mathcal{H}}_0^1(X) \oplus \bar{\mathcal{H}}_0^1(X_\infty), \text{ and for } \quad \varphi \in \mathcal{H}_0^1(X), \\ Q^N(X_0)(\varphi,\varphi) + Q^N(X_\infty)(\varphi,\varphi) = Q(H)(\varphi,\varphi). \end{gathered}$$

If A and B are self adjoint operators defined on a Hilbert space $\mathcal{L}$ and $Q(A)$ and $Q(B)$ are their corresponding quadratic forms with domains $\mathcal{D}(Q(A))$ and $\mathcal{D}(Q(B))$

$$Q(A)(\varphi,\varphi) \geq M||\phi||, \quad \phi \in \mathcal{D}(Q(A)) \text{ and } Q(B)(\psi,\psi) \geq M||\psi||, \quad \psi \in \mathcal{D}(Q(B)),$$

we say that

$$A \leq B \text{ if } \mathcal{D}(Q(B)) \subset \mathcal{D}(Q(A)) \text{ and } Q(A)(\varphi,\varphi) \leq Q(B)(\varphi,\varphi), \varphi \in \mathcal{D}(Q(B)).$$

This translates into the following

Proposition 3.1. *Let X_0, X_∞, $\mathcal{H}^D(X_\bullet)$ and $\mathcal{H}^N(X_\bullet)$, $\bullet = 0, \infty$, be defined as above, then*

$$\Upsilon_0^N \oplus \Upsilon_\infty^N \leq H \leq \Upsilon_0^D \oplus \Upsilon_\infty^D. \tag{3.5}$$

It follows from (3.4) that if $E > 0$,

$$N_E(\Upsilon_0^N) + N_E(\Upsilon_\infty^N) \leq N_E(H) \leq N_E(\Upsilon_0^D) + N_E(\Upsilon_\infty^D). \tag{3.6}$$

Since Υ_0^D and Υ_0^N are Schrödinger operators with C^∞ potentials on compact manifolds with boundary, their spectra satisfy (1.18). We will show that the point spectra of both Υ_∞^D and Υ_∞^N are either finite or infinite. In the latter case, we will show that both have the same asymptotic behavior as E goes to zero, and so this gives the asymptotic behavior of $N_E(H)$ as $E \to 0$.

3.1. Model Operators on X_∞.

Our analysis will be restricted to a small enough collar neighborhood of $\partial X = \{x = 0\}$ and we pick x to be a boundary defining function such that (1.4) holds for g and so Δ_g is given by (1.5) and the corresponding quadratic form for H on X_∞ with Dirichlet or Neumann boundary conditions defined in (3.3) is given by

$$\begin{gathered} Q^\bullet(X_\infty)(\varphi,\psi) = \int_0^{x_0}\int_{\partial X} \left((x\partial_x\varphi)(x\partial_x\bar{\psi}) + h^{jk}(x)(x\partial_{y_j}\varphi)(x\partial_{y_k}\bar{\psi}) + (V - \frac{n^2}{4})\varphi\bar{\psi} \right) \frac{\sqrt{h(x)}}{x^{n+1}} dydx, \\ \varphi,\psi \in \mathcal{W}_0^1(X_\infty), \text{ if } \bullet = D, \quad \varphi,\psi \in \bar{\mathcal{H}}_0^1(X_\infty), \text{ if } \bullet = N. \end{gathered}$$

Recall that by assumption

$$V(x,y) = -V_0(x) + O(V_1(x)), \quad V_0(x) > 0 \text{ and } V_1(x) > 0,$$

with V_0 and $V_1(x)$ satisfying either (1.7), (1.9) or (1.15), and $\rho = -\log x$. Since $h_{jk}(x) = h_{jk}(0) + x\tilde{h}_{jk}(x)$ and $h(0)$ is positive definite, then for x_0 small there exists $\gamma > 0$ and $a > 0$ such that

$$\begin{gathered}
Q^\bullet(X_\infty)(\varphi,\varphi) \leq \mathcal{Q}_+^\bullet(X_\infty)(\varphi,\varphi) \stackrel{\text{def}}{=} \\
\int_0^{x_0}\int_{\partial X}\left(|x\partial_x\varphi|^2 + h^{jk}(0)(x\partial_{y_j}\varphi)(x\partial_{y_k}\bar{\varphi})\right)(1+\gamma x)\frac{\sqrt{h(0)}}{x^{n+1}}dydx \\
\int_0^{x_0}\int_{\partial X}\left(-V_0(x) + aV_1(x) + xW_1^+(x) - \frac{n^2}{4}\right)|\varphi|^2(1+\gamma x)^{\frac{n+1}{n-1}}\frac{\sqrt{h(0)}}{x^{n+1}}dydx, \\
xW_1^+(x) = (-V_0 + aV_1 - \frac{n^2}{4})\frac{1}{(1+\gamma x)^{\frac{n+1}{n-1}}}\left(1 - (1+\gamma x)^{\frac{n+1}{n-1}}\right), \\
\varphi \in \mathcal{W}_0^1(X_\infty),\ \text{if } \bullet = D,\quad \varphi \in \bar{\mathcal{H}}_0^1(X_\infty),\ \text{if } \bullet = N.
\end{gathered}$$

and

$$\begin{gathered}
Q^\bullet(X_\infty)(\varphi,\varphi) \geq \mathcal{Q}_-^\bullet(X_\infty)(\varphi,\varphi) \stackrel{\text{def}}{=} \\
\int_0^{x_0}\int_{\partial X}\left(|x\partial_x\varphi|^2 + h^{jk}(0)(x\partial_{y_j}\varphi)(x\partial_{y_k}\bar{\varphi})\right)(1-\gamma x)^{\frac{n+1}{n-1}}\frac{\sqrt{h(0)}}{x^{n+1}}dydx \\
\int_0^{x_0}\int_{\partial X}\left(-V_0(x) - aV_1(x) + xW_1^-(x) - \frac{n^2}{4}\right)|\varphi|^2(1-\gamma x)^{\frac{n+1}{n-1}}\frac{\sqrt{h(0)}}{x^{n+1}}dydx, \\
xW_1^-(x) = (-V_0 - aV_1 - \frac{n^2}{4})\frac{1}{(1-\gamma x)^{\frac{n+1}{n-1}}}\left(1 - (1-\gamma x)^{\frac{n+1}{n-1}}\right), \\
\varphi \in \mathcal{W}_0^1(X_\infty),\ \text{if } \bullet = D,\quad \varphi \in \bar{\mathcal{H}}_0^1(X_\infty),\ \text{if } \bullet = N.
\end{gathered}$$

Notice that the quadratic forms $\mathcal{Q}_\pm^\bullet(X_\infty)$ correspond to the operators

$$\begin{aligned}
\mathcal{M}_+ &= \Delta_{g_+} + V_0(x) + aV_1(x) + xW_1^+(x) - \frac{n^2}{4} \text{ and} \\
\mathcal{M}_- &= \Delta_{g_-} + V_0(x) - aV_1(x) + xW_1^-(x) - \frac{n^2}{4},
\end{aligned}$$

where

$$\begin{aligned}
g_+ &= (1+\gamma x)^{\frac{2}{n-1}}(\frac{dx^2}{x^2} + \frac{h(0)}{x^2}) \text{ and} \\
g_- &= (1-\gamma x)^{\frac{2}{n-1}}(\frac{dx^2}{x^2} + \frac{h(0)}{x^2}).
\end{aligned} \tag{3.7}$$

We define $\mathcal{M}_\pm^\bullet$, $\bullet = N, D$ to be the corresponding self adjoint operators with Dirichlet or Neumann boundary conditions. To emphasize the difference, we will use the notation $\mathcal{Q}_\pm^\bullet(X_\infty) = \mathcal{Q}^\bullet(\mathcal{M}_\pm)$, $\bullet = D, N$. Therefore we have shown that

$$\begin{aligned}
&\mathcal{Q}^D(\mathcal{M}_-)(\varphi,\varphi) \leq Q^D(X_\infty)(\varphi,\varphi) \leq \mathcal{Q}^D(\mathcal{M}_+)(\varphi,\varphi),\quad \varphi \in \mathcal{W}_0^1(X_\infty), \\
&\mathcal{Q}^N(\mathcal{M}_-)(\varphi,\varphi) \leq Q^N(X_\infty)(\varphi,\varphi) \leq \mathcal{Q}^N(\mathcal{M}_+)(\varphi,\varphi),\quad \varphi \in \bar{\mathcal{H}}_0^1(X_\infty).
\end{aligned} \tag{3.8}$$

Notice that the $L^2(X_\infty)$ spaces defined with respect to g or $g_\pm$ are the same, with different but equivalent norms, and there are constants $C_j^\pm$, $j=1,2$ such that

$$\begin{aligned} C_1^-||\varphi||_{L^2_{g_-}} &\leq ||\varphi||_{L^2_g} \leq C_2^-||\varphi||_{L^2_{g_-}}, \\ C_1^+||\varphi||_{L^2_{g_+}} &\leq ||\varphi||_{L^2_g} \leq C_2^+||\varphi||_{L^2_{g_+}}. \end{aligned} \tag{3.9}$$

So the quadratic forms $Q^\bullet(\mathcal{M}_\pm)$ and $Q^\bullet(X_\infty)$, $\bullet = D, N$, have the same domain, but are defined with respect to different norms. However, (3.8) makes sense.

We also remark that one may extend the metrics $g_\pm$ to the manifold X, so that it becomes an AHM, and as a consequence of Theorem 2.2 we obtain

$$\sigma_{ess}(\mathcal{M}_\pm^\bullet) = [0,\infty), \quad \bullet = D, N. \tag{3.10}$$

We now appeal to the following characterization of the eigenvalues of a self-adjoint operator, see for example page 1543 of [7] or Theorem 3 of [26]:

Theorem 3.2. *Let H be a separable Hilbert space with inner product $\langle u, v\rangle$ and let A be a self adjoint operator corresponding to a semi-bounded quadratic form Q with domain $\mathcal{D}(Q)$. Suppose that the essential spectrum of A, $\sigma_{ess}(A) = [0,\infty)$, and its point spectrum satisfies*

$$\sigma_{pp}(A) = \{\lambda_1 \leq \lambda_2 \leq \ldots\}.$$

For $u \in \mathcal{D}(Q)$, $u \neq 0$, let $R(u) = \frac{Q(u,u)}{\langle u,u\rangle}$ denote the Rayleigh quotient, and for $n \in \mathbb{N}$, let

$$\mu_n = \inf\{\max\{R(u), u \in \mathcal{B} \text{ such that } \mathcal{B} \subset \mathcal{D}(Q) \text{ is a subspace } \dim \mathcal{B} = n\}\}.$$

Then $\mu_n \leq 0$. If $\mu_n = 0$ then A has at most $n-1$ eigenvalues $\lambda_j < 0$, counted with multiplicity. If $\mu_n < 0$, then $\mu_n = \lambda_n$, is the n-th eigenvalue of A counted with multiplicity.

The important aspect of this characterization is that there is no orthogonality required, as would be the case if we switched the order of max and min .

Recall that the domains of these quadratic forms satisfy

$$\begin{aligned} \mathcal{D}(\mathcal{Q}^D(\mathcal{M}_\pm)) &= \mathcal{D}(Q^D(X_\infty)) = \mathcal{W}_0^1(X_\infty) \times \mathcal{W}_0^1(X_\infty) \text{ and} \\ \mathcal{D}(\mathcal{Q}^N(\mathcal{M}_\pm)) &= \mathcal{D}(Q^N(X_\infty)) = \bar{\mathcal{H}}_0^1(X_\infty) \times \bar{\mathcal{H}}_0^1(X_\infty), \end{aligned}$$

and if we put together (3.8) and (3.9) we obtain

$$\begin{aligned} \frac{1}{C_2^-}\frac{\mathcal{Q}^D(\mathcal{M}_-)(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{L^2_{g_-}}} &\leq \frac{Q^D(X_\infty)(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{L^2_g}} \leq \frac{1}{C_1^+}\frac{\mathcal{Q}^D(\mathcal{M}_+)(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{L^2_{g_+}}}, \\ \frac{1}{C_2^-}\frac{\mathcal{Q}^N(\mathcal{M}_-)(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{L^2_{g_-}}} &\leq \frac{Q^N(X_\infty)(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{L^2_g}} \leq \frac{1}{C_1^+}\frac{\mathcal{Q}^N(\mathcal{M}_+)(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{L^2_{g_+}}}, \end{aligned}$$

and as a consequence of Theorem 2.2 and Theorem 3.2 and (3.10) we arrive at the following

Proposition 3.3. *Let $\lambda_j(\Upsilon_\infty^D)$ and $\lambda_j(\mathcal{M}_\pm^D)$ denote the eigenvalues of the operators with Dirichlet boundary conditions. Similarly, let $\mu_j(\Upsilon_\infty^N)$ and $\mu_j(\mathcal{M}_\pm^N)$ denote the eigenvalues of the operators with Neumann boundary conditions. If $\mathcal{M}_-^\bullet$ has finitely many eigenvalues, so do $\mathcal{M}_+^\bullet$ and $\Upsilon_\infty^\bullet$, $\bullet = D, N$. If $\mathcal{M}_+^\bullet$ has infinitely many eigenvalues, so do $\mathcal{M}_-^\bullet$ and $\Upsilon_\infty^\bullet$, $\bullet = D, N$ and if $C_j^\pm$, $j=1,2$,* *are as defined in* (3.9)*, then for all* j*,*

$$\begin{aligned}\frac{1}{C_2^-}\lambda_j(\mathcal{M}_-^D) \leq \lambda_j(\Upsilon_\infty^D) \leq \frac{1}{C_1^+}\lambda_j(\mathcal{M}_+^D),\\ \frac{1}{C_2^-}\mu_j(\mathcal{M}_-^N) \leq \mu_j(\Upsilon_\infty^N) \leq \frac{1}{C_1^+}\mu_j(\mathcal{M}_+^N).\end{aligned} \tag{3.11}$$

In particular, in the case there exist infinitely many eigenvalues, then for all $E<0$*,*

$$N_{C_2^- E}(\mathcal{M}_-^\bullet) \leq N_E(\Upsilon_\infty^\bullet) \leq N_{C_1^+ E}(\mathcal{M}_+^\bullet), \quad \bullet = D, N \tag{3.12}$$

4. The Asymptotic Behavior of $N_E(\mathcal{M}_\pm^\bullet)$, $\bullet = D, N$ as $E \to 0$

We will show that under the hypotheses of Theorems 1.1, 1.2 and 1.4, both $\mathcal{M}_\pm^D$ and $\mathcal{M}_\pm^N$ finitely many negative eigenvalues, or both have infinitely many, and in the latter case, depending on the potential V_0, the counting functions $N_E(\mathcal{M}_\pm^\bullet)$, $\bullet = D, N$, have the same asymptotic behavior $E \to 0$, given by Theorems 1.1, 1.2 and 1.4. More precisely, we will prove the following

Proposition 4.1. *Let* $\mathcal{M}_\pm^\bullet$, $\bullet = D, N$*, be defined as above and let* $N_E(\mathcal{M}_\pm^\bullet)$ *denote the corresponding counting function of eigenvalues. We have the following:*

T.1 Suppose $V_0(e^{-\rho})$ *and* $V_1(e^{-\rho})$ *satisfy* (1.8)*. If* ρ_0 *is large enough and* $\delta < 0$*, then* $\mathcal{M}_\pm^\bullet$ *has no negative eigenvalues, but if* $\delta \in (0,2)$*, then* $N_E(\mathcal{M}_\pm^\bullet)$ *satisfies* (1.8)*.*

T.2 Suppose $V_0(e^{-\rho})$ *and* $V_1(e^{-\rho})$ *satisfy* (1.9)*. If* ρ_0 *is large enough and* $c < \frac{1}{4}$*, then* $\mathcal{M}_\pm^\bullet$ *has no negative eigenvalues, but if* $c > \frac{1}{4}$*, then* $N_E(\mathcal{M}_\pm^\bullet)$ *satisfies* (1.10)*.*

T.3 Suppose $V_0(e^{-\rho})$ *and* $V_1(e^{-\rho})$ *satisfy* 1.15*. If* ρ_0 *is large enough and* $c_N < \frac{1}{4}$*, then* $\mathcal{M}_\pm^\bullet$ *has no negative eigenvalues, but if* $c_N > \frac{1}{4}$ *then* $N_E(\mathcal{M}_\pm^\bullet)$ *satisfies* (1.16)*.*

These results, together with equations (3.12), (3.6) and (1.18) respectively prove Theorems 1.1, 1.2 and 1.4.

We will consider the Dirichlet and Neumann eigenvalue problems in $X_\infty = \{x \leq x_0\}$ for the operators $\mathcal{M}_\pm$ defined above. We will drop the $\pm$ sub-indices and work with $\gamma, a \in \mathbb{R}$. We will assume that x_0 is small enough so that $x|\gamma| < \frac{1}{2}$ for $x \leq x_0$. We will work with the metric

$$\mathcal{G} = (1+\gamma x)^{\frac{2}{n-1}}\left(\frac{dx^2}{x^2} + \frac{h(0)}{x^2}\right).$$

We find that

$$\Delta_\mathcal{G} = -x^{n+1} f^{-n-1}\partial_x\left(f^{n-1}x^{1-n}\partial_x\right) + x^2 f^{-1}\Delta_{h(0)}, \quad f(x) = (1+\gamma x)^{\frac{2}{n-1}}.$$

To get rid of the factor $\frac{n^2}{4}$ in $\mathcal{M}_\pm^\bullet$, we observe that

$$x^{-\frac{n}{2}}(\Delta_\mathcal{G} - \frac{n^2}{4})x^{\frac{n}{2}} = -f^{-n-1}x\partial_x(f^{n-1}x\partial_x) + x^2 f^{-2}\Delta_{h(0)} - \frac{n}{2}xA, \quad A = f^{-n-1}\partial_x f^{n-1}.$$

So we want to study the eigenvalue problems

$$\begin{gathered}\left(-f^{-2n}(f^{n-1}x\partial_x)^2 + x^2 f^{-2}\Delta_{h(0)} - \frac{n}{2}xA - V_0(x) + aV_1(x) + x\mathcal{W}(x) + E\right)u^\bullet = 0, \quad E > 0, \bullet = D, N,\\ u^D(x_0, y) = 0 \text{ or } \partial_x u^N(x_0, y) = 0.\end{gathered}$$

We multiply the equation by f^{2n}, and use that $f^{2n}(x) = 1 + x\widetilde{f}(x)$, and we arrive at

$$\begin{gathered}\Big(-(f^{n-1}x\partial_x)^2 + x^2 f^{2(n-1)}\Delta_{h(0)} - V_0(x) + aV_1(x) + x\widetilde{\mathcal{W}}(x) + E\Big)\, u^\bullet = 0, \quad E>0, \bullet = D.N \\ u^D(x_0,y) = 0 \text{ or } \partial_x u^N(x_0,y) = 0.\end{gathered}$$

and we define r to be such that

$$\frac{dr}{r} = \frac{dx}{x(1+\gamma x)^2}, \quad r = 0 \text{ if } x = 0,$$

and we find that

$$x = r + r^2 X(r), \quad X \in C^\infty([0,1)), \tag{4.1}$$

and the equation becomes

$$\begin{gathered}\Big(-(r\partial_r)^2 + r^2(1+rF(r))\Delta_{h(0)} - V_0(r) + aV_1(r) + r\widetilde{\mathcal{W}}(r) + E\Big)\, u^\bullet = 0, \quad E>0, \bullet = D,N \\ u^D(r_0,y) = 0 \text{ or } \partial_r u^N(r_0,y) = 0.\end{gathered}$$

Notice that $\widetilde{\mathcal{W}}(r)$ will depend on E, but only with a term of the form $Ev(r)$ with v bounded. Finally, we set $r = e^{-\rho}$, $\rho \in [\rho_0, \infty)$, and the equation becomes

$$\begin{gathered}\mathcal{M}u^\bullet = -Eu^\bullet, \quad \bullet = D,N \quad u^D(\rho_0,y) = 0 \text{ or } \partial_\rho u^N(\rho_0,y) = 0, \text{ where} \\ \mathcal{M} = -\partial_\rho^2 + q(\rho)e^{-2\rho}\Delta_{h(0)} - V_0(e^{-\rho}) + aV_1(e^{-\rho}) + e^{-\rho}\widetilde{\mathcal{W}}(\rho), \quad q(\rho) = 1 + e^{-\rho}B(\rho).\end{gathered} \tag{4.2}$$

We decompose $u^\bullet(\rho,y)$ in Fourier series with respect to the eigenfunctions of $\Delta_{h(0)}$:

$$\begin{gathered}u^\bullet(\rho,y) = \sum_{j=0}^\infty u_j^\bullet(\rho)\psi_j(y), \quad u_j^\bullet(\rho) = \langle u^\bullet(\rho,y), \psi_j(y)\rangle_{L^2(\partial X, h(0))} \text{ where} \\ \Delta_{h(0)}\psi_j = \mu_j\psi_j, \quad 0 = \mu_0 < \mu_1 < \mu_2 \leq \mu_3 \dots\end{gathered} \tag{4.3}$$

Let

$$\begin{aligned}&U_j = \text{ the eigenspace corresponding to } \mu_j \text{ and define} \\ &m(\mu_j) = \dim U_j = \text{ multiplicity of } \mu_j,\end{aligned} \tag{4.4}$$

and so we have that

$$\begin{gathered}L^2(X_\infty) = \bigoplus_{j=1}^\infty U_j, \text{ and } \mathcal{M}^\bullet = \bigoplus_{j=1}^\infty \mathcal{M}_j^\bullet, \bullet = D,N, \text{ where} \\ \mathcal{M}_j = -\left(\frac{d}{d\rho}\right)^2 + e^{-2\rho}q(\rho)\mu_j - V_0(e^{-\rho}) + aV_1(e^{-\rho}) + e^{-\rho}\widetilde{\mathcal{W}}(\rho),\end{gathered} \tag{4.5}$$

For each j, $\mathcal{M}_j^\bullet$, $\bullet = D,N$, are self-adjoint operators, and it is well known that $\sigma_{ess}(\mathcal{M}_j^\bullet) = [0,\infty)$, see for example [8]. We prove in Appendix A that they have no eigenvalues in $[0,\infty)$, so they have only negative eigenvalues which only possibly accumulate at zero. It also follows from (4.5) that if $E > 0$,

$$\dim P_{(-\infty,-E)}(\mathcal{M}^\bullet) = \sum_{j=1}^\infty m(\mu_j) \dim P_{(-\infty,-E)}(\mathcal{M}_j^\bullet), \quad \bullet = D,N,$$

or in other words,

$$N_E(\mathcal{M}^\bullet) = \sum_{j=1}^{\infty} m(\mu_j) N_E(\mathcal{M}_j^\bullet), \quad \bullet = D, N, \tag{4.6}$$

where as above, $N_E(\mathcal{A})$ denotes the number eigenvalues of the operator $\mathcal{A}$ which are less than $-E$, counted with multiplicity.

The eigenfunctions $u_j^\bullet(\rho, E)(\rho)$, $\bullet = D, N$, with eigenvalue $-E$ satisfy

$$\begin{gathered}(\mathcal{M}_j^\bullet + E) u_j^\bullet = 0, \\ u_j^D(\rho_0, E) = 0 \text{ or } \partial_\rho u_j^N(\rho_0, E) = 0,\end{gathered} \tag{4.7}$$

For $E > 0$, we will consider the Cauchy problems

$$\begin{gathered}(\mathcal{M}_j^\bullet + E) u_j^\bullet(\rho, E) = 0, \quad \bullet = D, N, \\ u_j^D(\rho_0, E) = 0, \ \ \partial_\rho u_j^D(\rho_0, E) = 1 \text{ or } u_j^N(\rho_0, E) = 1, \ \ \partial_\rho u_j^N(\rho_0, E) = 0\end{gathered} \tag{4.8}$$

which have unique solutions in X_∞. Notice that $u_j^\bullet(\rho, E)$ exist for every E and are not necessarily eigenfunctions. In fact, it follows from Theorem A.2 in Appendix A that $u_j^\bullet(\rho, E)$ is an eigenfunction if and only if $u_j^\bullet(\rho, E) \sim Ce^{-\rho\sqrt{E}}$ as $\rho \to \infty$.

The key point of the proof Proposition 4.1 is

Proposition 4.2. *Let $u_j^\bullet(\rho, E)$, $\bullet = D, N$ be the unique solutions of the corresponding Cauchy problems in (4.8). Let $Z_j^\bullet(E)$ denote the number of its zeros, which are different from $\rho = \rho_0$ in the case $\bullet = D$. If $E < 0$, then*

$$N_E(\mathcal{M}_j^\bullet) = Z_j^\bullet(E), \quad \bullet = D, N, \tag{4.9}$$

and as a consequence of (4.6) we have

$$N_E(\mathcal{M}^\bullet) = \sum_{j=1}^{\infty} m(\mu_j) Z_j^\bullet(E), \quad \bullet = D, N. \tag{4.10}$$

This result is somewhat well known and its proof is essentially, but not quite, the same as the proof of Theorem XIII.8 of [24] and for the convenience of the reader, we provide the details in Appendix B.

4.1. **The Set up of the Problems.** Now we have to count the zeros of solutions of (4.8). We set up the general type of problem for V as in Theorems 1.2, 1.1 and 1.4. The arguments we use do not depend on the boundary condition, so we work with the Dirichlet problem in (4.8). We consider the problem

$$\begin{gathered}\left(-\frac{d^2}{d\rho^2} - V_0(e^{-\rho}) + e^{-2\rho}(1 + B(\rho)e^{-\rho})\mu + E + aV_1(e^{-\rho}) + e^{-\rho}\widetilde{\mathcal{W}}(\rho)\right) u = 0, \quad E > 0, \\ u(\rho_0) = 0, \quad \partial_\rho u(\rho_0) = 1,\end{gathered} \tag{4.11}$$

where $B(\rho), \widetilde{\mathcal{W}}(\rho) \in C^\infty([\rho_0, \infty)) \cap L^\infty([\rho_0, \infty))$.

We will denote

$$\begin{gathered}\mathcal{R}(\rho) = aV_1(\rho) + e^{-\rho} G(\rho), \text{ and} \\ \mathcal{P}(\rho) = \mu e^{-2\rho}(1 + e^{-\rho} B(\rho)) + E,\end{gathered} \tag{4.12}$$

and (4.11) is reduced to

$$\left(-\frac{d^2}{d\rho^2} - V_0(e^{-\rho}) + \mathcal{P}(\rho) + \mathcal{R}(\rho)\right) u = 0, \quad E > 0, \\ u(\rho_0) = 0, \tag{4.13}$$

We will deal with each case (1.7), (1.9) and in general (1.15) separately.

4.2. **Proof of Item T.1 of Proposition 4.1.** In this case $V_0(e^{-\rho}) = c\rho^{-2+\delta}$. We multiply the equation by $\rho^{2-\delta}$, and notice that

$$\rho^{-\frac{1}{2}(1-\frac{\delta}{2})}(\rho^{2-\delta}\frac{d^2}{d\rho^2})\rho^{\frac{1}{2}(1-\frac{\delta}{2})} = (\rho^{1-\frac{\delta}{2}}\frac{d}{d\rho})^2 - \frac{1}{4}(1-\frac{\delta^2}{4})\rho^{-\delta}.$$

So if $u(\rho) = \rho^{\frac{1}{2}(1-\frac{\delta}{2})}w(\rho)$, then (4.11) becomes

$$\left(-(\rho^{1-\frac{\delta}{2}}\frac{d}{d\rho})^2 - c + \frac{1}{4}(1-\frac{\delta^2}{4})\rho^{-\delta} + \mathcal{E}(\rho)\right) w = 0, \\ w(\rho_0) = 0, \text{ where} \\ \mathcal{E}(\rho) = \rho^{2-\delta}(\mathcal{P}(\rho) + \mathcal{R}(\rho)) \tag{4.14}$$

Set $t = \frac{2}{|\delta|}\sqrt{c}\,\rho^{\frac{\delta}{2}}$, and in this case (4.14) becomes

$$\left(-\frac{d^2}{dt^2} - 1 + \frac{1}{c}\mathcal{E}(\rho(t)) + \frac{1}{4c}(1-\frac{\delta^2}{4})\left(\frac{t|\delta|}{2\sqrt{c}}\right)^{-2}\right) w = 0, \quad t = \sqrt{c}\,\frac{2}{|\delta|}\rho^{\frac{\delta}{2}}, \\ w(t_0) = 0. \tag{4.15}$$

The case $\delta < 0$: Recall that by assumption,

$$V_1(e^{-\rho}) = \rho^{2-\delta}(\log\rho)^{-\varepsilon}, \quad \varepsilon > 0,$$

and therefore $\rho^{2-\delta}\mathcal{R}(\rho) \to 0$ as $\rho \to \infty$. Since $\mathcal{P}(\rho) > 0$, there exists $t_0 > 0$ independent of E and μ such that for $t < t_0$,

$$U(t) = -1 + \frac{1}{c}\mathcal{E}(\rho(t)) + \frac{1}{4c}(1-\frac{\delta^2}{4})\left(\frac{t|\delta|}{2\sqrt{c}}\right)^{-2} > \\ -1 + \frac{1}{4c}\rho^{2-\delta}\mathcal{R}(\rho) + \frac{1}{4}(1-\frac{\delta^2}{4})\left(\frac{t|\delta|}{2\sqrt{c}}\right)^{-2} > 0,$$

and so $w(t)$ is a solution of a differential equation

$$\frac{d^2w}{dt^2} = U(t)w, \ t < t_0, \text{ with } U(t) > 0, \\ w(t_0) = 0 \tag{4.16}$$

We have three possibilities for $w'(t_0)$: $w'(t_0) = 0$, or $w'(t_0) > 0$ or $w'(t_0) < 0$. If $w'(t_0) = 0$, then by uniqueness, $w(t) = 0$ for $t < t_0$. If $w'(t_0) > 0$, since $w(t)$ is C^∞, there exists $t_1 < t_0$ such that $w'(t) > 0$ for $t_1 < t \leq t_0$ and so $w(t) < 0$ for $t_1 < t \leq t_0$ and therefore, $w''(t) < 0$ for $t_1 < t \leq t_0$, and so, $w'(t) < w'(t_0) < 0$ for $t_1 < t < t_0$ and $w(t) < 0$ for $t_1 < t < t_0$. Repeating this argument we conclude that $w(t) < 0$ for all $t < t_0$. If $w'(t_0) < 0$, since $-w(t)$ also solves the equation, then $w(t) > 0$ for all $t < t_0$. Therefore we conclude that either $w(t) = 0$ for all $t > t_0$ or $w(t)$ has no zeros for $t > t_0$.

In this case we conclude from (4.10) that for this choice of ρ_0,

$$N_E(\mathcal{M}^\bullet) = 0, \quad \bullet = N, D. \tag{4.17}$$

The case $0 < \delta < 2$: In this case, in view of the discussion above, the set of zeros of the solution w is contained in the set

$$\{\rho > \rho_0 : -1 + \frac{1}{c}\mathcal{E}_1(\rho(t)) \leq 0\}, \text{ where } \mathcal{E}_1(\rho) = \rho^{2-\delta}(\mathcal{P} + \mathcal{R} + \frac{1}{4}(1 - \frac{\delta^2}{4})\rho^{-2}).$$

Recall that

$$\mathcal{R}(\rho) = aV_1(e^{-\rho}) + e^{-\rho}G(\rho), \text{ with } V_1(e^{-\rho}) = \rho^{-2+\delta}(\log\rho)^{-\varepsilon}, \quad \varepsilon > 0,$$

and so there exists ρ_0, independent of μ and E, such that

$$\left| \frac{1}{c}\rho^{2-\delta}\mathcal{R}(\rho) + \frac{1}{4c}(1 - \frac{\delta^2}{4})\rho^{-\delta} \right| \leq \frac{1}{2} \text{ for all } \rho > \rho_0. \tag{4.18}$$

Therefore, for this choice of ρ_0,

$$\{\rho \geq \rho_0 : \frac{1}{c}\mathcal{E}_1(\rho) \leq \frac{3}{2}\} \subset \{\rho \geq \rho_0 : \frac{\rho^{2-\delta}}{c}(E + \mu e^{-2\rho}(1 + B(\rho)e^{-\rho})) \leq 2\},$$

and for ρ_0 large

$$\begin{aligned} &\{\rho \geq \rho_0 : \frac{\rho^{2-\delta}}{c}(E + \mu e^{-2\rho}(1 + B(\rho)e^{-\rho})) \leq 2\} \subset \{\rho \geq \rho_0 : \frac{\rho^{2-\delta}}{c}(E + \frac{1}{2}\mu e^{-2\rho}) \leq 2\} \subset \\ &\{\rho \geq \rho_0 : \frac{E}{c}\rho^{2-\delta} \leq 2\} \cap \{\rho \geq \rho_0 : \frac{\mu}{2c}\rho^{2-\delta}e^{-2\rho} \leq 2\}. \end{aligned} \tag{4.19}$$

Also, if $\rho_0 > 4$, the set $\{\rho \geq \rho_0 : \frac{1}{c}\rho^{2-\delta}(E + \frac{1}{2}\mu e^{-2\rho}) \leq 2\}$ is either empty or

$$\{\rho \geq \rho_0 : \frac{\rho^{2-\delta}}{c}(E + \frac{1}{2}\mu e^{-2\rho}) \leq 2\} = [\rho_1, \rho_2], \quad \rho_0 \leq \rho_1 < \rho_2,$$

and this is because if $F(\rho) = \rho^{2-\delta}(E + \frac{1}{2}\mu e^{-2\rho})$, then

$$F'(\rho) + F''(\rho) = (2-\delta)\rho^{-\delta}(E + \frac{1}{2}\mu e^{-2\rho})(\rho + 1 - \delta) + 4\mu\rho^{1-\delta}e^{-2\rho}(\rho - 2(2-\delta)),$$

and therefore, if $F'(\rho) = 0$, then $F''(\rho) > 0$, so if $F(\rho)$ has a critical point, it is a local minimum.

So if we take ρ_2 such that

$$\frac{E}{c}\rho_2^{2-\delta} = 2, \text{ so } \rho_2 = \left(\frac{2c}{E}\right)^{\frac{1}{2-\delta}},$$

and since $\rho^{2-\delta}$ is an increasing function, then

$$\frac{E}{c}\rho^{2-\delta} \leq 2 \text{ for } \rho \leq \rho_2.$$

But in view of (4.19) if $\rho \leq \rho_2$, $\frac{1}{c}\rho^{2-\delta}(E + \frac{1}{2}\mu e^{-2\rho}) \leq 2$, only if μ satisfies

$$\mu \leq \frac{4c}{\rho^{2-\delta}}e^{2\rho} \leq \frac{4c}{\rho_2^{2-\delta}}e^{2\rho_2} = 2Ee^{\left(\frac{2c}{E}\right)^{\frac{1}{2-\delta}}} \stackrel{\text{def}}{=} \mu_{\delta,U}$$

then

$$\frac{1}{c}\mathcal{E}_1(\rho) \leq \frac{3}{2} \text{ provided } \rho_0 \leq \rho \leq \rho_2 \text{ and } \mu \leq \mu_{\delta,U}.$$

As usual, see for example Chapter 8 of [4], to count the zeros of w one sets

$$\theta(t) = \tan^{-1}\left(\frac{w(t)}{w'(t)}\right), \text{ where } w(t) \text{ satisfies (4.15).}$$

The number of zeros of w in an interval $[t_1, t_2]$, coincide with the μ number of times $\theta(t) = k\pi$, for some $k \in \mathbb{N}$. But it follows from (4.15) that

$$\frac{d\theta}{dt} = \frac{(w')^2 - w''w}{w^2 + (w')^2} = 1 - \frac{1}{c}\mathcal{E}_1(t(\rho))\left(\frac{w^2}{w^2 + (w')^2}\right), \tag{4.20}$$

and since $\frac{1}{c}\mathcal{E}_1(t(\rho)) < \frac{3}{2}$, it follows that $\left|\frac{d\theta}{dt}\right| \leq 3$, and so

$$\theta(t_2) - \theta(t_1) \leq 3(t_2 - t_1) \leq 3t_2, \quad t_2 = \frac{2\sqrt{c}}{\delta}\rho_2^{\frac{\delta}{2}}. \tag{4.21}$$

Therefore if $Z(w)$ denotes the number of zeros of $w(t)$, the $Z(w) \leq \frac{3t_2}{\pi}$. But

$$t_2 = \frac{2\sqrt{c}}{\delta}\rho_2^{\frac{\delta}{2}} = \frac{2\sqrt{c}}{\delta}\left(\frac{2}{cE}\right)^{\frac{\delta}{2(2-\delta)}},$$

and so we conclude that for E small

$$Z_j^\bullet(E) \leq \frac{6\sqrt{c}}{\pi\delta}\left(\frac{2}{cE}\right)^{\frac{\delta}{2(2-\delta)}} \text{ for all } j,$$

and so in view of (4.10), as $E \to 0$, we have for $\bullet = D, N$,

$$N_E(\mathcal{M}^\bullet) \leq \sum_{\mu_j \leq \mu_{\delta,U}} m_j(\mu_j) Z_j^\bullet(E) \leq C\left(\frac{2}{cE}\right)^{\frac{\delta}{2(2-\delta)}} \sum_{\mu \leq \mu_{\delta,U}} m_j(\mu_j) = C\left(\frac{2}{cE}\right)^{\frac{\delta}{2(2-\delta)}} \mathcal{N}_{\mu_{\delta,U}}(\Delta_{h(0)}),$$

where $\mathcal{N}_\kappa(\Delta_{h(0)})$, is the number of eigenvalues of $\Delta_{h(0)}$ which are less than or equal to κ counted with multiplicity. Weyl's Law, see Corollary 17.5.8 of [12], says that

$$\mathcal{N}_\kappa(\Delta_{h(0)}) = C_n\kappa^n + O(\kappa^{n-1}), \tag{4.22}$$

and this implies that

$$N_E(\mathcal{M}^\bullet) \leq C\left(\frac{2}{cE}\right)^{\frac{\delta}{2(2-\delta)}} {\mu_{\delta,U}}^n = C\left(\frac{2}{cE}\right)^{\frac{\delta}{2(2-\delta)}} \left(Ee^{\left(\frac{2}{cE}\right)^{\frac{1}{2-\delta}}}\right)^n,$$

and we find that

$$\log\log N_E(\mathcal{M}^\bullet) \leq \frac{1}{2-\delta}\log E^{-1} + O(1), \text{ as } E \to 0,$$

which is the upper bound of (1.8).

To obtain a similar lower bound, we will find an interval $[\rho_1, \rho_2]$, such that $\rho_1 > \rho_0$, with ρ_0 as in (4.18), and

$$\frac{\rho^{2-\delta}}{c}(E + \frac{1}{2}\mu e^{-2\rho}) \leq \frac{1}{4} \text{ for } \rho \in [\rho_1, \rho_2].$$

In that case it follows that $\frac{1}{c}\mathcal{E}_1(\rho) \leq \frac{3}{4}$ and we deduce from (4.20) that

$$\theta(t_2) - \theta(t_1) \geq \frac{1}{4}(t_2 - t_1),$$

and so

$$Z_j^\bullet(E) \geq \frac{1}{4\pi}(t_2 - t_1), \quad t_j = \frac{2\sqrt{c}}{\delta}\rho_j^{\frac{\delta}{2}}, \quad j = 1, 2.$$

Notice that, since $E\rho^{2-\delta}$ is an increasing function and $\mu\rho^{2-\delta}e^{-2\rho}$ is an decreasing function, we choose ρ_2 such that

$$\frac{E}{c}\rho_2^{2-\delta}=\frac{1}{8},\quad \rho_2=\left(\frac{c}{8E}\right)^{\frac{1}{2-\delta}},$$

and for $\rho_1=\frac{1}{2}\rho_2$, we only pick μ such that

$$\frac{\mu}{2c}\rho_1^{2-\delta}e^{-2\rho_1}\leq\frac{1}{8},$$

but this implies that

$$\mu\leq\frac{c}{4\rho_1^{2-\delta}}e^{2\rho_1}\leq\frac{c}{4\rho_2^{2-\delta}}e^{2\rho_2}=2Ee^{\left(\frac{c}{8E}\right)^{\frac{1}{2-\delta}}}\stackrel{\text{def}}{=}\mu_{\delta,L}.$$

Therefore, for this choice of ρ_1 and ρ_2, and $\mu\leq\mu_{\delta,L}$ it follows that $\frac{1}{c}\rho^{2-\delta}(\frac{1}{2}\mu e^{-2\rho}+E)\leq\frac{1}{4}$ in $[\frac{1}{2}\rho_2,\rho_2]$. Therefore,

$$Z_j^{\bullet}(E)\geq\frac{1}{4\pi}(t_2-t_1)=\frac{2\sqrt{c}}{\pi\delta}\left(\rho_2^{\frac{\delta}{2}}-\rho_1^{\frac{\delta}{2}}\right)\geq\frac{2\sqrt{c}}{\pi\delta}\rho_2^{\frac{\delta}{2}}\left(1-(\frac{1}{2})^{\frac{\delta}{2}}\right)\geq C\left(\frac{c}{8E}\right)^{\frac{\delta}{2-\delta}}$$

It follows from (4.10) that as $E\to0$,

$$N_E(\mathcal{M}^{\bullet})\geq\sum_{\mu_j\leq\mu_{\delta,L}}m_j(\mu_j)Z_j^{\bullet}(E)\geq C\left(\frac{c}{4E}\right)^{\frac{1}{2-\delta}}(\mu_{\delta,L})^n\geq C\left(\frac{c}{8E}\right)^{\frac{\delta}{2-\delta}}E^ne^{n\left(\frac{c}{8E}\right)^{\frac{1}{2-\delta}}},$$

and this shows that

$$\log\log N_E(\mathcal{M}^{\bullet})\geq\frac{1}{2-\delta}\log E^{-1}+O(1)\text{ as }E\to0.$$

This ends the proof of item T.1 of Proposition 4.1 and together with equations (3.12), (3.6) and (1.18) it also ends the proof of Theorem 1.1.

4.3. **The Proof of Item T.2 of Proposition 4.1.** In this case $V_0(e^{-\rho})=c\rho^{-2}$, $V_1(e^{-\rho})=\rho^{-2}(\log\rho)^{-\varepsilon}$, and

$$\begin{gathered}\left(-\frac{d^2}{d\rho^2}-c\rho^{-2}+\mathcal{R}(\rho)+\mathcal{P}(\rho)\right)u=0,\quad E>0,\\ u(\rho_0)=0,\end{gathered}\tag{4.23}$$

where $\mathcal{R}$ and $\mathcal{P}$ are given by (4.12). Next we multiply the equation by ρ^2, set $u=\rho^{\frac{1}{2}}w$ and notice that

$$\rho^{-\frac{1}{2}}(\rho^2\frac{d}{d\rho})^2\rho^{\frac{1}{2}}=(\rho\frac{d}{d\rho})^2-\frac{1}{4},\tag{4.24}$$

then (4.23) becomes

$$\begin{gathered}-(\rho\frac{d}{d\rho})^2w-(c-\frac{1}{4})w+\mathcal{E}(\rho)w=0,\quad \mathcal{E}(\rho)=\rho^2(\mathcal{R}(\rho)+\mathcal{P}(\rho))\\ w(\rho_0)=0.\end{gathered}\tag{4.25}$$

where

Set $s = \log \rho$, and (4.25) becomes

$$\left(-\frac{d^2}{ds^2} - (c - \frac{1}{4}) + \mathcal{E}(\rho(s))\right) w = 0, \quad (4.26)$$
$$w(s_0) = 0, \text{ where } s_0 = \log(\rho_0).$$

The case $c < \frac{1}{4}$: Since $\mathcal{P}(\rho) > 0$, and $\rho^2\mathcal{R}(\rho) \to 0$ as $\rho \to \infty$, there exists $R > 0$ independent of μ and E such that

$$(\frac{1}{4} - c) + \mathcal{E}(\rho) \geq (\frac{1}{4} - c) + \rho\mathcal{R}(\rho) > 0, \text{ for } \rho_0 > R.$$

Therefore, we have an equation as (4.16), with $U(t) = (\frac{1}{4} - c) + \mathcal{E}(\rho(s)) > 0$ and so w has no zeros, and again we conclude from (4.10) that for this choice of ρ_0, (4.17) holds.

The case $c > \frac{1}{4}$: We set $t = \lambda s$, $\lambda = (c - \frac{1}{4})^{\frac{1}{2}}$ and (4.26) becomes

$$\left(-\frac{d^2}{dt^2} - 1 + \frac{1}{\lambda^2}\mathcal{E}(\rho(t))\right) w = 0, \quad (4.27)$$
$$w(t_0) = 0, \text{ where } t_0 = \lambda \log(\rho_0),$$

The argument used above shows that the zeros of the solution w of (4.26) lie on the set

$$\{t > t_0 : \frac{\rho(t)^2}{\lambda^2}(\mathcal{R}(\rho(t)) + \mathcal{P}(\rho(t))) \leq 1\}.$$

We pick ρ_0 large so that $|\rho^2\mathcal{R}(\rho)| < \frac{1}{2}$, for $\rho > \rho_0$, and therefore,

$$\{\rho \geq \rho_0 : \frac{\rho^2}{\lambda^2}(\mathcal{R}(\rho) + \mathcal{P}(\rho)) \leq \frac{3}{2}\} \subset \{\rho \geq \rho_0 : \frac{\rho^2}{\lambda^2}(E + \frac{1}{2}\mu e^{-2\rho}) \leq 2\}.$$

But we have

Lemma 4.3. *Let $f(\rho) > 0$ be a C^∞ function such that $f''(\rho) > 0$ and $f(\rho) > f'(\rho)$ for $\rho > \rho_0$. Then the function $F(\rho) = f(\rho)(E + \mu e^{-2\rho})$ is convex and therefore the set*

$$\Omega(E, \mu, C) = \{t > t_0 : f(\rho)(E + \frac{1}{2}\mu e^{-2\rho}) \leq C\}$$

is either empty or is equal to an interval $[\rho_1, \rho_2]$ with $\rho_1 \geq \rho_0$.

Proof. We find that

$$F''(\rho) = f''(\rho)(E + \frac{1}{2}\mu e^{-2\rho}) + 4\mu(f(\rho) - f'(\rho))e^{-2\rho} > 0.$$

□

Therefore, since $\rho = \rho(\rho)$,

$$\{\rho \geq \rho_0 : \frac{\rho^2}{\lambda^2}(E + \frac{1}{2}\mu e^{-2\rho}) \leq 2\} = [\rho_1, \rho_2],$$

but we also observe that

$$\{\rho \geq \rho_0 : \frac{\rho^2}{\lambda^2}(E + \frac{1}{2}\mu e^{-2\rho}) \leq 2\} \subset \{\rho \geq \rho_0 : \frac{E}{\lambda^2}\rho^2 \leq 2\} \cap \{\rho \geq \rho_0 : \frac{\mu}{2\lambda^2}\rho^2 e^{-2\rho} \leq 2\}. \quad (4.28)$$

As above one sets

$$\theta(\rho) = \tan^{-1}\left(\frac{w(\rho)}{w'(\rho)}\right), \text{ where } w(\rho) \text{ satisfies (4.27)}$$

and then (4.20) holds, and since $\frac{1}{\lambda^2}(E+\frac{1}{2}\mu e^{-2\rho}) \leq 2$, it follows that $\left|\frac{d\theta}{d\rho}\right| \leq 3$ and so

$$\theta(t_2) - \theta(t_1) \leq 3(t_2 - t_1) \leq 3t_2. \tag{4.29}$$

Therefore if $Z(w)$ denotes the number of zeros of $w(\rho)$ we have

$$Z(w) = \frac{2}{\pi}(\theta(t_2) - \theta(t_1)) \leq \frac{3t_2}{\pi}.$$

It follows from (4.28) that if ρ_2 is such that

$$\frac{E}{\lambda^2}\rho_2^2 = 2,$$

then $\frac{E}{\lambda_2^2}\rho^2 \leq 2$, for $\rho \leq \rho_2$. So we conclude that for E small

$$Z_j^\bullet(E) \leq \frac{\lambda}{\pi}\log(\sqrt{\frac{\lambda^2}{E}}) \leq C\log E^{-\frac{1}{2}} \text{ as } E \to 0$$

But on the other hand, according to (4.28) we must have

$$\mu \leq \frac{4\lambda^2}{\rho^2}e^{2\rho} \leq \frac{4\lambda^2}{\rho_2^2}e^{2\rho_2} = 2Ee^{2\sqrt{\frac{2\lambda_2^2}{E}}} \stackrel{\text{def}}{=} \mu_U,$$

and so $\frac{\rho^2}{\lambda^2}(E+\frac{1}{2}e^{-2\rho}\mu\} \leq 2$ on an interval $[\rho_0, \rho_2]$. and in view of (4.10), as $E \to 0$, we have

$$N_E(\mathcal{M}^\bullet) \leq \sum_{\mu_j \leq \mu_U} m_j(\mu_j) Z_j^\bullet(E) \leq C\log E^{-\frac{1}{2}} \sum_{\mu_j \leq \mu_U} m_j(\mu_j) = C\log E^{-\frac{1}{2}}\mathcal{N}_{\mu_U}(\Delta_{h(0)}),$$

and because of (4.22) this implies that

$$N_E(\mathcal{M}^\bullet) \leq C\mu_U^n \log E^{-\frac{1}{2}}.$$

and it follows from the definition of μ_U that

$$\log\log N_E(H) \leq -\frac{1}{2}\log E + O(1)$$

which gives the upper bound in (1.10).

To prove a similar lower bound for $N_E(H)$ will find an interval $[\rho_1, \rho_2]$ such that

$$\frac{\rho^2}{\lambda^2}(E + \frac{1}{2}\mu e^{-2\rho}) \leq 2 \text{ on } [\rho_1, \rho_2],$$

and so it follows from (4.20) that $\frac{d\theta}{dt} \geq \frac{1}{4}$ and so $\theta(t_2) - \theta(t_1) \geq \frac{1}{4}(t_2 - t_1)$ and

$$Z(\omega) \geq \frac{1}{4\pi}(t_2 - t_1), \quad t_j = \lambda \log \rho_j, \quad j = 1, 2.$$

Since $E\rho^2$ is increasing we pick ρ_2 such that

$$\frac{E}{\lambda^2}\rho_2^2 = \frac{1}{8}, \tag{4.30}$$

and so $\frac{E}{\lambda^2}\rho^2 \leq \frac{1}{8}$, for $\rho \leq \rho_2$. Pick ρ_1 such that $\rho_1 = \frac{1}{M}\rho_2$, with M to be chosen, and we want

$$\frac{\mu}{2\lambda^2}\rho_1^2 e^{-2\rho_1} \leq \frac{1}{8},$$

thus

$$\mu \leq \frac{\lambda^2}{4\rho_1^2} e^{2\rho_1} \leq \frac{\lambda^2}{4\rho_2^2} e^{2\rho_2} = 2Ee^{2\sqrt{\frac{\lambda^2}{8E}}} \stackrel{\text{def}}{=} \mu_{_L}. \tag{4.31}$$

Since $t_1 = \lambda \log \rho_1$ and $t_2 = \lambda \log \rho_2$, we can choose M, independent of E and μ, such that

$$Z(w) \geq \frac{1}{4\pi}(t_2 - t_1) \geq \frac{\lambda}{4\pi} \log\left(\frac{\rho_2}{\rho_1}\right) = \frac{\lambda}{4\pi} \log M > 1.$$

Again we conclude from (4.10) that for $\mu_{_L}$ as in (4.31),

$$N_E(\mathcal{M}^\bullet) \geq \sum_{\mu_j \leq \mu_{_L}} m_j(\mu_j) Z_j^\bullet(E) + O(1) \geq C \sum_{\mu_j \leq \mu_{_L}} m_j(\mu_j) = C\mu_{_L}^n.$$

This gives that

$$\log\log N_E(\mathcal{M}^\bullet) \geq -\frac{1}{2}\log E + O(1),$$

which is the lower bound in (1.10) and proves item T.2 of Proposition 4.1 and together with equations (3.12), (3.6) and (1.18) it also ends the proof of Theorem 1.2.

4.4. **The Proof of Item T.3 of Proposition 4.1.** We first consider the case $N = 1$ in (1.15) and we have equation (4.13) with

$$V_0(e^{-\rho}) = \frac{1}{4}\rho^{-2} + c_1\rho^{-2}(\log\rho)^{-2} \text{ and } V_1(e^{-\rho}) = \rho^{-2}(\log\rho)^{-2}(\log\rho)^{-\varepsilon}.$$

We multiply the equation by ρ^2, set $u = \rho^{\frac{1}{2}} w$ and use (4.24) and we obtain

$$\begin{aligned} &\left(-(\rho\partial_\rho)^2 - c_1(\log\rho)^{-2} + \rho^2\mathcal{E}(\rho)\right) w = 0, \quad \mathcal{E}(\rho) = \mathcal{R}(\rho) + \mathcal{P}(\rho). \\ &w(\rho_0) = 0. \end{aligned} \tag{4.32}$$

Now we set $\xi = \log\rho$, multiply (4.32) by ξ^2 and set $w = \xi^{\frac{1}{2}} v$, use (4.24) and we obtain

$$\begin{aligned} &\left(-(\xi\frac{d}{d\xi})^2 - (c_1 - \frac{1}{4}) + \mathcal{E}_1(\rho(\xi))\right) v = 0, \quad \mathcal{E}_1(\rho) = \rho^2(\log\rho)^2\mathcal{E}(\rho), \\ &v(\xi_0) = 0, \end{aligned}$$

As before, if $c_1 < \frac{1}{4}$, v has no zeros for $\xi > \xi_0$, and so it follows from (4.10) that $\mathcal{M}^\bullet$, $\bullet = N, D$ have no eigenvalues.

If $c_1 > \frac{1}{4}$, set $\lambda_1 = (c_1 - \frac{1}{4})^{\frac{1}{2}}$ and set $\tau = \lambda_1 \log\xi$, and we obtain

$$\begin{aligned} &\left(-\partial_\tau^2 - 1 + \frac{1}{\lambda_1^2}\mathcal{E}_1(\tau(\rho))\right) v = 0, \quad \tau = \lambda_1 \log_{(2)}\rho, \\ &v(\tau_0) = 0. \end{aligned}$$

As before, we assume that ρ_0 is large enough so that

$$\left|\frac{1}{\lambda_1}^2 \rho^2(\log\rho)^2\mathcal{R}(\rho)\right| | \leq \frac{1}{2},$$

and for this choice of ρ_0, consider the set

$$\{\rho \geq \rho_0 : \frac{1}{\lambda_1^2}(\rho \log \rho)^2(\mathcal{R}(\rho) + \mathcal{P}(\rho)) \leq \frac{3}{2}\} \subset \{\rho \geq \rho_0 : \frac{1}{\lambda_1^2}(\rho \log \rho)^2 \mathcal{R}(\rho) \leq 2\} \subset$$
$$\{\rho \geq \rho_0 : \frac{E}{\lambda_1^2}(\rho \log \rho)^2 \leq 2\} \cap \{\rho \geq \rho_0 : \frac{\mu}{2\lambda_1^2}(\rho \log \rho)^2 e^{-2\rho} \leq 2\}.$$

Since $f(\rho) = \frac{1}{\mathcal{G}_1(\rho)} = \rho^2(\log \rho)^2$ satisfies the hypothesis of Lemma 4.3, we deduce that

$$\{\rho \geq \rho_0 : \frac{1}{\lambda_1^2}(\rho \log \rho)^2(E + \frac{1}{2}\mu e^{-2\rho}) \leq 2\} = [\rho_1, \rho_2],$$

but then we must have

$$\frac{E}{\lambda_1^2}(\rho_2 \log \rho_2)^2 \leq 2$$

and in particular if we take

$$R = \frac{2}{\log A_1} A_1 \text{ where } A_1 = \sqrt{\frac{2\lambda_1^2}{E}},$$

then, for E small, and independently of μ,

$$R \log R = 2A_1(1 + \frac{\log 2}{\log A_1} - \frac{1}{\log A_1}\log(\log A_1)) \geq A_1,$$

and so $\rho_2 < R$ and therefore for $\rho \leq R$, we must have

$$\mu \leq \frac{4\lambda_1^2}{(\rho \log \rho)^2} e^{2\rho} \leq \frac{4\lambda_1^2}{(R \log R)^2} e^{2R} \stackrel{\text{def}}{=} \mu_{U_1}. \tag{4.33}$$

Since $\tau = \lambda_1 \log \log \rho$, we obtain for small E,

$$\theta(\tau_2) - \theta(\tau_1) \leq 3(\tau_2 - \tau_1) \leq 3\tau_2 = 3\lambda_1 \log \log \rho_2 \leq 3\lambda_1 \log \log R \leq C\lambda_1 \log \log E^{-1},$$

and for μ_{U_1} as in (4.33),

$$N_E(\mathcal{M}^\bullet) \leq C\lambda_1 \log \log E^{-1} \mathcal{N}_{\partial X, h_0}(\mu_{U_1}) \leq C\lambda_1 \log(\log E^{-1})(\mu_{U_1})^n.$$

It follows that

$$\log_{(3)} N_E(H) \leq \frac{1}{2}\log_{(2)} E^{-1} + O(1),$$

and this gives the upper bound in (1.14).

To obtain the lower bound we find ρ_1 and ρ_2 such that

$$\frac{\rho^2(\log \rho)^2}{\lambda_1^2}(\frac{1}{2}\mu e^{-2\rho} + E) \leq \frac{1}{4} \text{ for } \rho \in [\rho_1, \rho_2],$$

and this can be achieved if we take ρ_1 and ρ_2 such that

$$\frac{E}{\lambda_1^2}(\rho_2 \log \rho_2)^2 \leq \frac{1}{8}, \quad \rho_1 = \rho_2^{\frac{1}{M}}, \quad \text{with } M \text{ large to be chosen independently of } \mu, E.$$

For instance, for E small enough, take

$$\rho_2 = \frac{1}{\log \beta_1}\beta_1, \text{ where } \beta_1 = \sqrt{\frac{\lambda_1^2}{8E}},$$

But we also want $\frac{\mu}{2\lambda_1^2}\rho^2(\log \rho)^2 e^{-2\rho} \leq \frac{1}{8}$ and so we need μ to satisfy

$$\mu \leq \frac{1}{16(\rho_1 \log \rho_1)^2} e^{2\rho_1} \stackrel{\text{def}}{=} \mu_{L_1}.$$

Since $\tau = \lambda_1 \log\log \rho$, we can choose M, independent of E and μ, so that

$$\theta(\tau_2) - \theta(\tau_1) \geq \frac{1}{4\pi}(\tau_2 - \tau_1) \geq \frac{\lambda_1}{4\pi} \log\left(\frac{\log \rho_2}{\log \rho_1}\right) = \frac{\lambda_1}{4\pi} \log(M) > 1.$$

This implies that for μ_{L1} as above

$$N_E(\mathcal{M}^\bullet) \geq \mathcal{N}_{\partial X, h_0}(\mu_{L_1}) \geq C(\mu_{L_1})^n,$$

and we conclude that

$$\log_{(3)} N_E(\mathcal{M}^\bullet) \geq \log_{(2)} E^{-1} + O(1),$$

which implies (1.14).

Next, we consider the case $N = 2$ and

$$\begin{aligned} V_0(\rho) &= \frac{1}{4}\rho^{-2} + \frac{1}{4}\rho^{-2}(\log \rho)^{-2} + c_2 \rho^{-2}(\log\rho)^{-2}(\log_{(2)}\rho)^{-2}, \\ V_1(\rho) &= \rho^{-2}(\log\rho)^{-2}(\log_{(2)}\rho)^{-2}(\log\rho)^{-\varepsilon}, \end{aligned}$$

This time we set $\eta = \log_{(3)} \rho$ and we obtain, with

$$\begin{gathered} \left(-(\frac{d}{d\eta})^2 - (c_2 - \frac{1}{4}) + \mathcal{E}_2(\rho(\eta))\right) v = 0, \text{ where} \\ \mathcal{E}_2(\rho) = \rho^2(\log\rho)^2(\log\log\rho)^2(\mathcal{R}(\rho) + \mathcal{P}(\rho)), \\ v(\eta_0) = 0. \end{gathered}$$

As before, if $c_2 < \frac{1}{4}$, v has no zeros for $\mu > \mu_0$, and so $\mathcal{M}^\bullet$ has no negative eigenvalues. If $c_2 > \frac{1}{4}$, set $\lambda_2 = (c_2 - \frac{1}{4})^{\frac{1}{2}}$ and set $\tau = \lambda_2 \eta$, and we obtain

$$\begin{gathered} \left(-(\frac{d}{d\tau})^2 - 1 + \frac{1}{\lambda_2^2}\mathcal{E}_2(\tau(\rho))\right) v = 0, \\ v(\tau_0) = 0, \end{gathered}$$

We pick ρ_0 large such that

$$\left|\frac{1}{\lambda_2^2}\rho^2(\log\rho)^2(\log\log\rho)^2\mathcal{R}(\rho)\right| \leq \frac{1}{2}, \quad \rho \geq \rho_0,$$

and for $\mathcal{G}_2 = (\rho^2(\log\rho)^2(\log\log\rho)^2)^{-1}$, we consider the set

$$\{\rho \geq \rho_0 : \frac{1}{\lambda_2^2 \mathcal{G}_2(\rho)}(E + \frac{1}{2}\mu e^{-2\rho}) \leq 2\} \subset \{\rho \geq \rho_0 : \frac{E}{\lambda_2^2\mathcal{G}_2(\rho)} \leq 2\} \cap \{\rho \geq \rho_0 : \frac{\mu}{2\lambda_2^2 \mathcal{G}_2(\rho)} e^{-2\rho} \leq 2\}.$$

and since $f(\rho) = \frac{1}{\mathcal{G}_2(\rho)} = \rho^2(\log\rho)^2(\log\log\rho)^2$ satisfies the hypothesis of Lemma 4.3, we find that

$$\{\rho \geq \rho_0 : \frac{1}{\lambda_2^2\mathcal{G}_2(\rho)}(E + \frac{1}{2}\mu e^{-2\rho}) \leq 2\} = [\rho_1, \rho_2],$$

and so we must have

$$\frac{E}{\lambda_2^2 \mathcal{G}_2(\rho_2)} \leq 2 \text{ and so } \rho(\log\rho)(\log\log\rho) \leq \sqrt{\frac{2\lambda_2^2}{E}} = A_2.$$

If we take $\hat{R} = \frac{2A_2}{\log A_2(\log\log A_2)}$, then

$$\hat{R}(\log \hat{R})(\log\log \hat{R}) = 2A_2(1+o(1)), \text{ as } E \to 0,$$

and $\rho_2 \leq \hat{R}$, and so for $\rho \leq \rho_2$ we must have

$$\mu \leq 4\lambda_2^2 \mathcal{G}_2(\rho) e^{2\rho} \leq 4\lambda_2^2 \mathcal{G}_2(\hat{R}) e^{2\hat{R}} \stackrel{\text{def}}{=} \mu_{U_2}.$$

It follows from (4.10) that

$$N_E(\mathcal{M}^\bullet) \leq C(\log_{(3)} \hat{R})(\mu_{U2})^n,$$

which implies that

$$\log_{(4)} N_E(\mathcal{M}^\bullet) \leq \log_{(3)} E^{-1} + O(1) \text{ as } E \to 0,$$

which is the upper bound of (1.16) when $N = 2$.

Again, to obtain the lower bound we find ρ_1 and ρ_2 such that

$$\frac{1}{\lambda_2^2 \mathcal{G}_2(\rho)}(E + \frac{1}{2}\mu e^{-2\rho}) \leq \frac{3}{4} \text{ for } \rho \in [\rho_1, \rho_2],$$

and we pick ρ_2 such that

$$\frac{E}{\lambda_2^2}(\rho_2(\log \rho_2)(\log\log \rho_2))^2 \leq \frac{1}{8}$$

and for instance we can just take

$$\rho_2 = \frac{1}{(\log \beta_2)(\log\log \beta_2)}\beta_2, \text{ where } \beta_2 = \sqrt{\frac{\lambda_2^2}{8E}}.$$

We then and pick ρ_1 such that $\log \rho_1 = (\log \rho_2)^{\frac{1}{M}}$, with M to be chosen, and we need to restrict the values of μ so that

$$\mu \leq \frac{\lambda_2^2}{4} \mathcal{G}_2(\rho_1) e^{2\rho_1} = \mu_{L_2}.$$

Since $\tau = \lambda_2 \log_{(3)} \rho$, we can choose M, independent of μ and E, such that

$$\theta(\tau_2) - \theta(\tau_1) \geq \frac{1}{4\pi}(\tau_2 - \tau_1) \geq \frac{C\lambda_2}{4\pi} \log\left(\frac{\log(\log \rho_2)}{\log(\log \rho_1)}\right) \geq \frac{C\lambda_2}{4\pi} \log M > 1.$$

This implies that for μ_{L_2} as above

$$N_E(\mathcal{M}^\bullet) \geq \mathcal{N}_{\partial X, h_0}(\kappa_{L2}) \geq C(\mu_{L_2})^n.$$

This implies that

$$\log_{(4)} N_E(\mathcal{M}^\bullet) \geq \log_{(3)} E^{-1} + O(1),$$

which is the lower bound (1.16) for the case $N = 2$.

The proof in the general case in (1.15) follows the same principle. In this case,

$$V_0(\rho) = \frac{1}{4}\rho^{-2} + \frac{1}{4}\sum_{j=1}^{N-1} \mathcal{G}_j(\rho) + c_N \mathcal{G}_N(\rho),$$
$$V_1(\rho) = \mathcal{G}_N(\rho)(\log \rho)^{-\varepsilon},$$

where $\mathcal{G}_j(\rho)$ is defined in (1.15). We pick ρ_0 such that

$$\log_{(j)} \rho > 1 \text{ for } \rho \geq \rho_0, \text{ provided } j \leq N. \tag{4.34}$$

This time we set $\eta = \log_{(N+1)} \rho$ and we obtain

$$\left(-(\frac{d}{d\eta})^2 - (c_N - \frac{1}{4}) + \mathcal{E}_N(\rho(\eta))\right) v = 0, \quad \mathcal{E}_N(\rho) = (\mathcal{G}_N(\rho))^{-2}(\mathcal{R}(\rho) + \mathcal{P}(\rho)),$$
$$v(\eta_0) = 0.$$

If $c_N < \frac{1}{4}$, then v has no zeros for $\eta > \eta_0$ if η_0 is large and so $\mathcal{M}^\bullet$ has no negative eigenvalues. If $c_N > \frac{1}{4}$, set $\lambda_N = (c_N - \frac{1}{4})^{\frac{1}{2}}$ and set $\tau = \lambda_N \eta$, and we obtain

$$\left(-(\frac{d}{d\tau})^2 - 1 + \frac{1}{\lambda_N^2}\mathcal{E}_N(\rho(\tau))\right) v = 0, \quad \tau = \lambda_N \log_{(N+1)} \rho,$$
$$v(\rho_0) = 0,$$

Again, we then follow the steps in the proof of the previous cases and pick ρ_0 such that

$$|\mathcal{G}_N(\rho)^{-2}\mathcal{R}(\rho)| \leq \frac{1}{2},$$

and so the zeros of v will be contained in the set consider the set

$$\left\{\rho \geq \rho_0 : \frac{1}{\lambda_N^2(\mathcal{G}_N(\rho))^2}(E + \frac{1}{2}\mu e^{-2\rho}) \leq 2\right\} \subset$$
$$\left\{\rho \geq \rho_0 : \frac{E}{\lambda_2^2(\mathcal{G}_N(\rho))^2} \leq 2\right\} \cap \left\{\rho \geq \rho_0 : \frac{\mu}{2\lambda_2^2(\mathcal{G}_N(\rho))^2}e^{-2\rho} \leq 2\right\}.$$

and since $\frac{1}{(\mathcal{G}_N(\rho))^2}$ satisfies the hypothesis of Lemma 4.3, we find that

$$\{\rho \geq \rho_0 : \frac{1}{\lambda_N^2(\mathcal{G}_N(\rho))^2}(E + \frac{1}{2}\mu e^{-2\rho}) \leq 2\} = [\rho_1, \rho_2],$$

and we pick ρ_2 such that

$$\frac{1}{\lambda_N^2(\mathcal{G}_N(\rho))^2}E = 2,$$

we have

$$\frac{1}{\mathcal{G}_N(\rho)} = \rho(\log\rho)(\log\log\rho)\dots(\log_{(N)}\rho) \leq \sqrt{\frac{2\lambda_N^2}{E}}, \quad \rho \in [\rho_0, \rho_2].$$

and as before, if we pick

$$R_N = \frac{2A_N}{\log A_N(\log\log A_N)\dots\log_{(N)} A_N}, \quad A_N = \sqrt{\frac{2\lambda_N^2}{E}},$$

then

$$\frac{1}{\mathcal{G}_N(R_N)} = R_N(\log R_N)\dots\log_{(N)} R_N \geq A_N,$$

and therefore,

$$\rho_2 \leq R_N.$$

We must also have μ such that

$$\mu \leq 4\lambda_N^2(\mathcal{G}_N(\rho))^2 e^{2\rho} = 4\mathcal{G}_N(R_N)e^{2R_N} \stackrel{\text{def}}{=} \mu_{U_N}.$$

But on the other hand,

$$\theta(\tau_2) - \theta(\tau_1) \leq \frac{3}{2\pi}\tau_2 \leq \frac{3}{2\pi}\lambda_N \log_{(N+1)} R_N,$$

and therefore

$$N_E(\mathcal{M}^\bullet) \le C(\log_{(N+1)} R_N)(\mu_{U_N})^n,$$

and this implies that

$$\log N_E(\mathcal{M}^\bullet) = R_N\left(2 + \frac{1}{R_N}\left(\log \mathcal{G}_N(R_N) + \log(C\log_{(N+1)} R_N)\right)\right),$$

and so

$$\log_{(2)} N_E(\mathcal{M}) = \log R_N(1 + O(1)) \text{ as } E \to 0$$

But

$$\log R_N = (\log E^{-1})(\frac{1}{2} + O(1)) \text{ as } E \to 0,$$

and so

$$\log_{(2)} N_E(\mathcal{M}) = (\log E^{-1})(C + O(1)) \text{ as } E \to 0,$$

and this implies that for any $j \ge 2$,

$$\log_{(j+1)} N_E(\mathcal{M}) \le (\log_{(j)} E^{-1}) + O(1).$$

The case $j = N+1$ gives (1.16).

To establish the lower bound in (1.16) we need to find ρ_1, ρ_2 such that

$$\frac{1}{\lambda_2^2 \mathcal{G}_N(\rho)}(\frac{1}{2}\mu e^{-2\rho} + E) \le \frac{3}{4} \text{ for } \rho \in [\rho_1, \rho_2].$$

We pick ρ_2 such that

$$\frac{E}{\lambda_2^2 \mathcal{G}_N(\rho_2)} \le \frac{1}{8},$$

and we can just take

$$\rho_2 = \frac{\beta_N}{(\log \beta_N)(\log\log \beta_N)\dots(\log_{(N)} \beta_N)}, \text{ where } \beta_N = \sqrt{\frac{\lambda_N^2}{8E}},$$

and we pick ρ_1 such that

$$\log_{(N-1)} \rho_1 = \left(\log_{(N-1)} \rho_2\right)^{\frac{1}{M}} \text{ with } M \text{ large enough, to be chosen independently of } \mu, E,$$

and only consider the values of μ such that

$$\mu \le \frac{\lambda_N^2}{4}\mathcal{G}_N(\rho_1)e^{2\rho_1} = \mu_{L_N}.$$

Since $\rho = \lambda_N \log_{(N+1)} \rho$, we can choose M, independent of E and μ, large enough such that

$$\theta(\rho_2) - \theta(\rho_1) \ge \frac{\lambda_N}{4\pi}(\log_{(N+1)} \rho_2 - \log_{(N+1)} \rho_1) = \frac{\lambda_N}{4\pi}\log\left(\frac{\log_{(N)} \rho_2}{\log_{(N)} \rho_1}\right) = \frac{\lambda_N}{4\pi}\log(M) > 1$$

In view of (4.10) and (4.22), this implies that

$$N_E(\mathcal{M}^\bullet) \ge C(\mu_{L_N})^n.$$

Then we have

$$\log N_E(\mathcal{M}^\bullet) \ge 2n\rho_1 + \log(C\mathcal{G}_N(\rho_1)) = \rho_1(2n + O(1)),$$

and so

$$\log_{(2)} N_E(\mathcal{M}^\bullet) = \log \rho_1 + O(1),$$

and we deduce that

$$\log_{N+1} N_E(\mathcal{M}^\bullet) = \log_{(N)} \rho_1 + O(1) = \frac{1}{M} \log_{(N)} \rho_2 + O(1),$$

and so

$$\log_{N+2} N_E(H) = \log_{(N+1)} \rho_2 + O(1).$$

But on the other hand

$$\log \rho_2 = (\log E^{-1})(\frac{1}{2} + O(1)),$$

and so

$$\log_{(j)} \rho_2 = \log_{(j)} E^{-1} + O(1), \quad j \geq 2,$$

This implies the lower bound in (1.16) and ends the proof of item T.3 of Proposition 4.1, and together with equations (3.12), (3.6) and (1.18) it also ends the proof of Theorem 1.4.

Appendix A. The spectrum of $\mathcal{M}_j$

Recall that $\mathcal{M}^\bullet$ is the operator defined in (4.2) with boundary conditions $\bullet = D, N$. We will show that $\mathcal{M}^\bullet$, $\bullet = D, N$ has no eigenvalues $E \geq 0$. If it did, then each $\mathcal{M}_j^\bullet$ defined in (4.8) would have the same eigenvalue. So we just need to show that $\mathcal{M}_j^\bullet$ has no eigenvalues in $[0, \infty)$. Recall that $V_0(\rho)$ satisfies the assumptions of either one of the Theorems 1.1, 1.2 or 1.4. We prove the following

Proposition A.1. *The operators $\mathcal{M}_j^\bullet$, $j \in \mathbb{N}$, $\bullet = D, N$, $j \in \mathbb{N}$, defined in (4.5) have no eigenvalues in $[0, \infty)$.*

Proof. If E is an eigenvalue of $\mathcal{M}_j^\bullet$, then there exists $\psi \in L^2((\rho_0, \infty))$ such that

$$\psi'' = \Big(-E + V_0(e^{-\rho}) - c_1 V_1(e^{-\rho}) + e^{-\rho} \widetilde{\mathcal{V}}(\rho)\Big) \psi \tag{A.1}$$

Now we appeal to Theorems 2.1 and 2.4 from Section 6.2 of [20], which we state in a single theorem:

Theorem A.2. *In a given finite or infinite interval (a_1, a_2), let $f(x)$ be a positive, twice continuously differentiable function, $g(\rho)$ a continuous real or complex function, and*

$$F(\rho) = \int [f^{-1/4}(f^{-1/4})'' - g f^{-1/2}] d\rho.$$

Then in this interval the differential equations

$$u''(\rho) = (f(\rho) + g(\rho)) u(\rho), \ \text{ and} \tag{A.2a}$$

$$w''(\rho) = (-f(\rho) + g(\rho)) w(\rho), \tag{A.2b}$$

have twice continuously differentiable solutions which in the case (A.2a) are given by

$$\begin{aligned} u_1(\rho) &= f^{-1/4}(\rho) e^{\int f^{1/2} d\rho}(1 + \varepsilon_1(\rho)), \\ u_2(x) &= f^{-1/4}(\rho) e^{-\int f^{1/2} d\rho}(1 + \varepsilon_2(\rho)), \end{aligned} \tag{A.3}$$

and in the case (A.2b) are given by

$$\begin{aligned} w_1(\rho) &= f^{-1/4}(\rho) e^{i\int f^{1/2} d\rho}(1 + \varepsilon_1(\rho)), \\ w_2(x) &= f^{-1/4}(\rho) e^{-i\int f^{1/2} d\rho}(1 + \varepsilon_2(\rho)), \end{aligned} \tag{A.4}$$

such that the error terms $\varepsilon_j(\rho)$, $j=1,2$ satisfy

$$\begin{gathered}|\varepsilon_1(\rho)| \leq e^{\frac{1}{2}\mathcal{V}_{a_1,\rho}(F)}-1 \text{ and } |\varepsilon_2(\rho)| \leq e^{\frac{1}{2}\mathcal{V}_{\rho,a_2}(F)}-1,\\ \frac{1}{2}f^{-1/2}(\rho)|\varepsilon_1'(x)| \leq e^{\frac{1}{2}\mathcal{V}_{a_1,\rho}(F)}-1 \text{ and } \frac{1}{2}f^{-1/2}(\rho)|\varepsilon_2'(x)| \leq e^{\frac{1}{2}\mathcal{V}_{\rho,a_2}(F)}-1,\end{gathered} \tag{A.5}$$

provided $\mathcal{V}_{a_1,\rho}(F)<\infty$. Here $\mathcal{V}_{a_1,\rho}(F)$ denotes the total variation of F on the interval (a_1,ρ).

We first show that one cannot have an eigenvalue $E>0$. We will consider the case of Theorem 1.1, $V_0(e^{-\rho})=c\rho^{-2+\delta}$ and $V_1(\rho)=\rho^{-2+\delta}(\log\rho)^{-\varepsilon}$, the other cases are very similar. We apply Theorem A.2 with

$$-f(\rho)=-E-c\rho^{-2+\delta}+c_1\rho^{-2+\delta}(\log\rho)^{-\varepsilon} \text{ and } g(\rho)=e^{-\rho}\mathcal{V}(\rho)$$

Then on the interval $[\rho_1,\infty)$, with ρ_1 large,

$$w_1(\rho)=f^{-\frac{1}{4}}e^{i\sqrt{f(\rho)}}(1+\varepsilon_1(\rho)),\quad w_2(\rho)=E^{-\frac{1}{4}}e^{-i\sqrt{f(\rho)}}(1+\varepsilon_2(\rho)).$$

But

$$(f^{-\frac{1}{4}})''f^{-\frac{1}{4}}=-\frac{c}{4}(3-\delta)(2-\delta)E^{-\frac{3}{2}}\rho^{-4+\delta}(1+o(1)) \text{ and } |g(\rho)|\leq Ce^{-\rho},$$

and so

$$\mathcal{V}_{\rho_1,\rho}(F)(\rho)=\int_{\rho_1}^{\rho}|F'(s)|ds\leq\int_{\rho_1}^{\rho}(M_1e^{-s}+M_2E^{-\frac{3}{2}}s^{-4+\delta})ds\leq C_1(e^{-\rho_1}+e^{-\rho})+C_2E^{-\frac{3}{2}}(\rho_1^{-3+\delta}+\rho^{-3+\delta})$$

and so

$$\varepsilon_1(\rho)\leq e^{\frac{1}{2}\mathcal{V}_{\rho_1,\rho}(F)(\rho)}-1\leq C((e^{-\rho_1}+e^{-\rho})+E^{-\frac{3}{2}}(\rho_1^{-3+\delta}+\rho^{-3+\delta})),\quad \delta\geq 0,$$

and hence for ρ_1 large, $1+\varepsilon_1(\rho)\sim c_1+c_2\rho^{-3+\delta}$ and therefore $w_1\notin L^2([\rho_1,\infty)$. A similar analysis works to estimate $\mathcal{E}_2$. Since there are constants C_1 and C_2 such that $\psi(E,\rho)=C_1w_1(\rho)+C_2w_2(\rho)$, it follows that $\psi\notin L^2([\rho_1,\infty))$ and so ψ cannot be an eigenfunction.

When $E=0$, and $\delta>0$, we apply the same argument with $-f=-c\rho^{-2+\delta}$ and we obtain

$$(f^{-\frac{1}{4}})''f^{-\frac{1}{4}}=c_1\rho^{-1-\frac{\delta}{2}}$$

and so we find that for ρ_1 large, $1+\varepsilon_j(\rho)\sim c_1+c_2\rho^{-\delta}$ and so there are no eigenfunctions with $E\geq 0$.

The last case does not quite apply when $\delta=0$ and we use an argument as in the proof of Hardy's inequality in [6]. We will prove the following

Lemma A.3. *Suppose $u\in L^2([\rho_0,\infty))$, $h(\rho)$ is continuous and $h(\rho)=o(1)$ as $\rho\to\infty$ and*

$$u''(\rho)=\rho^{-2}(-\frac{1}{4}+h(\rho))u \text{ in } (\rho_0,\infty),\quad c>0,$$

then $u(\rho)=0$ on $[\rho_0,\infty)$.

Proof. Since $u\in L^2([\rho_0,\infty))$, by using the equation and the Cauchy-Schwarz inequality, we find that $|u'(\rho)|\leq C\rho^{-\frac{3}{2}}$ and hence $|u(\rho)|\leq C\rho^{-\frac{1}{2}}$, and the equation gives $|u''(\rho)|\leq C\rho^{-\frac{-5}{2}}$. Therefore, if $\alpha\in(1,2)$, $\lambda=\frac{1}{2}(\alpha-1)>0$ and $\rho_1>\rho$,

$$\int_{\rho_1}^{\infty}\rho^{\alpha}(u'(\rho))^2d\rho=\int_{\rho_1}^{\infty}\rho^{\alpha}(\rho^{-\lambda}(\rho^{\lambda}u)'-\lambda\rho^{-1}u)^2d\rho\geq$$

$$\lambda^2\int_{\rho_1}^{\infty}\rho^{\alpha-2}(u(\rho))^2d\rho-\lambda\int_{\rho_1}^{\infty}\left((\rho^{\lambda}u)^2\right)'d\rho$$

and since $\lambda < \frac{1}{2}$, we deduce that for $\alpha \in (1,2)$,

$$\frac{(\alpha-1)^2}{4}\int_{\rho_1}^{\infty} \rho^{\alpha-2}(u(\rho))^2 d\rho \leq \int_{\rho_1}^{\infty} \rho^{\alpha}(u'(\rho))^2 d\rho.$$

We apply the same argument to the second derivative, and use that $|u'(\rho)| \leq C\rho^{-\frac{3}{2}}$, then for $\alpha \in (1,4)$,

$$\frac{(\alpha-1)^2}{4}\int_{\rho_1}^{\infty} \rho^{\alpha-2}(u'(\rho))^2 d\rho \leq \int_{\rho_1}^{\infty} \rho^{\alpha}(u''(\rho))^2 d\rho.$$

We combine these two estimates and we obtain, for $\alpha \in (3,4)$,

$$\frac{(\alpha-1)^2}{4}\frac{(\alpha-3)^2}{4}\int_{\rho_1}^{\infty} \rho^{\alpha-4}(u(\rho))^2 d\rho \leq \int_{\rho_1}^{\infty} \rho^{\alpha}(u''(\rho))^2 d\rho.$$

But the equation implies that

$$\frac{(\alpha-1)^2}{4}\frac{(\alpha-3)^2}{4}\int_{\rho_1}^{\infty} \rho^{\alpha-4}(u(\rho))^2 d\rho \leq \int_{\rho_1}^{\infty} (-\frac{1}{4}+h(\rho))^2\rho^{\alpha-4}(u(\rho))^2 \; d\rho.$$

Pick ρ_1 large and $\alpha = 4-\varepsilon$ with ε small and this implies that $u(\rho) = 0$ on $[\rho_1, \infty)$. Then $u = 0$ on (ρ_0, ∞) by uniqueness. □

□

Appendix B. The Proof of Proposition 4.2

We follow the arguments used in the proof of Theorem XIII.8 of [24]. We have already established that $\sigma_{ess}(\mathcal{M}_j^\bullet) = [0,\infty)$, $\bullet = D, N$ and that there are no eigenvalues in the essential spectrum. Then one needs to prove three lemmas:

Lemma B.1. *Let $V(\rho) \in C^\infty(I)$, $I \subset \mathbb{R}$ open and let $E \in \mathbb{R}$. Let $u(\rho, E)$, not identically zero, satisfy*

$$u''(\rho, E) = (V(\rho) - E)u(\rho, E) \text{ on } I.$$

If $a_0 = a_0(E_0) \in I$ is such that $u(a_0, E_0) = 0$. Then there exists $\delta > 0$ and a C^∞ function $a(E)$ defined for $|E - E_0| < \delta$ such that $a(E_0) = a_0$ and $u(a(E), E) = 0$.

Proof. We know from the existence and uniqueness and stability theorems for odes that $u(\rho, E)$ is a C^∞ function and since $u(\rho, E)$ is not identically zero, if $u(a_0, E_0) = 0$, then $\partial_\rho u(a_0, E_0) \neq 0$. The implicit function theorem then guarantees that there exists a C^∞ function $a(E)$ defined on an interval $|E - E_0| < \delta$ such that $a(E_0) = a_0$ and $u(a(E), E) = 0$. □

Lemma B.2. *As above, let $\bullet = D, N$. Let $\mathcal{M}_j$ be the operators defined in (4.8). Let V_0 and V_1 satisfy the hypotheses of either Theorem 1.1, 1.2 or 1.4. The following statements about $Z_j^\bullet(E)$ are true:*

1. *If $-E < 0$, then $Z_j^\bullet(E) < \infty$.*
2. *If $Z_j^\bullet(E_0) \geq m$, there exists $\delta > 0$ so that $Z_j^\bullet(E) \geq m$ for $|E - E_0| < \delta$.*
3. *$-E_0 < -E$, then $Z_j^\bullet(E) \geq Z_j^\bullet(E_0)$.*
4. *If $-E_0$ is an eigenvalue of $\mathcal{M}_j^\bullet$, and $-E_0 < -E$, then $Z_j^\bullet(E) \geq Z_j^\bullet(E_0) + 1$.*
5. *If $k > j$ and $\mu_k > \mu_j$, then $Z_j^\bullet(E) \geq Z_k^\bullet(E)$.*
6. *If $k > j$, $-E$ is an eigenvalue and $\mu_k > \mu_j$, then $Z_j^\bullet(E) \geq Z_k^\bullet(E) + 1$.*

Proof. We have already shown that item 1 is true. Lemma B.1 says that if $\rho_1 < \rho_2 < \ldots < \rho_{m-1} < \rho_m \in (\rho_0, \infty)$ are such that $u_j^\bullet(\rho_j, E_0) = 0$, then there exist $\delta > 0$ and C^∞ functions $r_j(E)$ defined in $|E - E_0| < \delta$ such that $r_j(E_0) = \rho_j$ and that $u_j^\bullet(r_j(E), E) = 0$, and therefore $Z_j^\bullet(E) \geq m$.

To prove item 3, we first consider the Dirichlet problem. This is the standard form of Sturm oscillation theorem. Let $\rho_0 < \rho_1 < \ldots < \rho_n$ be the zeros of $u_j^D(\rho, E_0)$. We claim that $u_j^D(\rho, E)$ has a zero in each of the intervals (ρ_j, ρ_{j+1}). To see that suppose that $u_j^D(\rho, E)$ does not have a zero in this interval. By possibly multiplying the functions by -1, we may assume that $u_j^D(\rho, E) > 0$ and $u_j^D(\rho, E_0) > 0$ in (ρ_j, ρ_{j+1}). In this case the $u_j'(\rho_j, E_0) > 0$ and $u_j'(\rho_{j+1}, E_0) < 0$. Therefore the integral

$$I^D = \int_{\rho_m}^{\rho_{m+1}} \left[(u_j^D)'(\rho, E_0) u_j^D(\rho, E) - u_j^D(\rho, E_0)(u_j^D)'(\rho, E) \right]' \, d\rho =$$
$$(u_j^D)'(\rho_{m+1}, E_0) u_j^D(\rho_{m+1}, E) - (u_j^D)'(\rho_m, E_0) u_j^D(\rho_m, E) \leq 0.$$

But on the other hand,

$$I^D = \int_{\rho_m}^{\rho_{m+1}} \left[(u_j^D)''(\rho, E_0) u_j^D(\rho, E) - u_j^D(\rho, E_0)(u_j^D)''(\rho, E) \right] \, d\rho =$$
$$(E_0 - E) \int_{\rho_m}^{\rho_{m+1}} u_j^D(\rho, E_0) u_j^D(\rho, E) \, d\rho > 0.$$

If E_0 is an eigenvalue, we claim that $u_j^D(\rho, E)$ also has a zero in (ρ_n, ∞). To see that we apply the same idea, but now one needs to justify the convergence of the integral from ρ_n to ∞. We appeal again to Theorem A.2. If we take $f = E_0$, and $f = E$, the solutions of (4.8), for ρ large are of the form

$$u_j^D(\rho, E_0) = (E_0)^{-\frac{1}{4}} (C_1 e^{-\rho\sqrt{E_0}}(1 + \varepsilon_1(\rho)) + C_2 e^{\rho\sqrt{E_0}}(1 + \varepsilon_2(\rho)),$$
$$u_j^D(\rho, E) = (E)^{-\frac{1}{4}} (\widetilde{C}_1 e^{-\rho\sqrt{E}}(1 + \varepsilon_1(\rho)) + \widetilde{C}_2 e^{\rho\sqrt{E}}(1 + \varepsilon_2(\rho)).$$

But since E_0 is an eigenvalue, $u_j^D(\rho, E_0) \in L^2((\rho_0, \infty))$ and $C_2 = 0$. Since $E_0 > E$, then integrals will involving terms of the type $e^{\rho(\sqrt{E} - \sqrt{E_0})} O(1)$ which will converge if $E_0 > E$.

As for the Neumann problem, the same argument applies with the exception of the interval (ρ_0, ρ_1). In this case, we know from the assumptions made in (4.8) that

$$(u_j^N)'(\rho_0, E_0) = (u_j^N)'(\rho_0, E) = 0, \quad u_j^N(\rho_0, E_0) = u_j^N(\rho_0, E) = 1,$$

and we also know that $u_j^N(\rho_1, E_0) = 0$. In this case we would have $u_j^N(\rho, E) > 0$ and $u_j^N(\rho, E_0) > 0$ in (ρ_0, ρ_1), and so we would have

$$(u_j^N)'(\rho_1, E_0) \leq 0 \text{ and } u_j^N(\rho_1, E) \geq 0$$

and therefore, the integral

$$I^N = \int_{\rho_0}^{\rho_1} \left[(u_j^N)'(\rho, E_0) u_j^N(\rho, E) - u_j^N(\rho, E_0)(u_j^N)'(\rho, E) \right]' \, d\rho = (u_j^N)'(\rho_1, E_0) u_j^N(\rho_1, E) \leq 0.$$

But as above,

$$I^N = (E_0 - E) \int_{\rho_0}^{\rho_1} u_j^N(\rho, E_0) u_j^N(\rho, E) \, d\rho > 0.$$

The same argument can be used to show that $Z_j^\bullet(E) \geq Z_k^\bullet(E)$, provided $j > k$ and $\mu_k > \mu_j$. In this case, we suppose that $\rho_1 < \rho_2 \ldots < \rho_n$ are the zeros of $u_k^D(\rho, E)$, and we want to show that

$u_j(\rho, D)$ has a zero in (ρ_m, ρ_{m+1}). We assume there are no zeros of $u_j(\rho, E)$ in (ρ_m, ρ_{m+1}) and we may assume that $u_k^D(\rho, E) > 0$ and $u_j^D(\rho, E) > 0$ on ρ_m, ρ_{m+1} and

$$(u_k^D)'(\rho_m, E) > 0, (u_k^D)'(\rho_{m+1}, E) < 0, \text{ and } u_j^D(\rho_m, E) \geq 0, u_j^D(\rho_{m+1}, E) \geq 0.$$

Then the integral

$$I^D = \int_{\rho_m}^{\rho_{m+1}} \left[(u_k^D)'(\rho, E) u_j^D(\rho, E) - u_k^D(\rho, E)(u_j^D)'(\rho, E) \right]' \, d\rho =$$
$$(u_k^D)'(\rho_{m+1}, E) u_j^D(\rho_{m+1}, E) - (u_k^D)'(\rho_m, E) u_j^D(\rho_m, E) \leq 0.$$

But on the other hand

$$I^D = \int_{\rho_m}^{\rho_{m+1}} (\mu_k - \mu_j) e^{-2\rho} u_k^D(\rho, E) u_j^D(\rho, E) \, d\rho > 0.$$

The same argument works for the Neumann problem and to prove item 6. □

Lemma B.3. *Let $-\lambda_{j,k}^\bullet$, $k = 1, 2, \ldots,$ denote the eigenvalues of $\mathcal{M}_j^\bullet$, $\bullet = D, N$. The following are true*

I. The eigenvalues have multiplicity one.
II. If $E \geq 0$, $m \in \mathbb{N}$ and $Z_j^\bullet(E) \geq m$, then $-\lambda_{j,m} < -E$. In particular

$$N_E(\mathcal{M}_j^\bullet) \geq Z_j^\bullet(E).$$

III. $Z_j^\bullet(\lambda_{j,k}^\bullet) = k - 1$.

Proof. The eigenvalues are simple by the uniqueness theorem for ordinary differential equations. By dividing an eigenfunction $\psi^\bullet(\rho)$ by a constant, one may assume it will satisfy $\psi^D(\rho_0) = 0$ and $(\psi^D)'(\rho_0) = 1$ or $(\psi^N)'(\rho_0) = 0$ and $\psi^N(\rho_0) = 1$, and one cannot have two different solutions with the same Cauchy data.

We will show that there exist at least m eigenvalues $\lambda_{j,k}^\bullet$ which are less than $-E$. Let $u_j^\bullet(\rho, E)$ be the solution of (4.8) and let $\rho_1 < \rho_2 < \ldots < \rho_M$, $M \geq m$, and $\rho_0 < \rho_1$, denote its zeros (not equal to ρ_0 in case $\bullet = D$) and let

$$\psi_k^\bullet(\rho) = \begin{cases} u_j^\bullet(\rho, E), \text{ if } \rho_k \leq \rho \leq \rho_{k+1}, \; k = 0, 1, \ldots, M-1 \\ \qquad 0 \text{ otherwise} \end{cases}$$

Obviously, $\langle \psi_i^\bullet, \psi_k^\bullet \rangle = 0$, if $i \neq k$. Let $\mathcal{U}$ be the M-dimensional subspace spanned by $\psi_k^\bullet$. If $\psi^\bullet = \sum_{j=1}^{m_0} a_k \psi_k^\bullet$, one can check that

$$\langle \mathcal{M}_j^\bullet \psi^\bullet, \psi^\bullet \rangle = -E \langle \psi^\bullet, \psi^\bullet \rangle,$$

it follows from the min-max principle, see Theorem XIII.2 of [24], that $-\lambda_{j,M}^\bullet \leq -E$ and in particular $\lambda_{j,m}^\bullet \leq -E$. But it follows from item 2 of Lemma B.2 that if $\varepsilon > 0$ is small enough, $Z_j^\bullet(E + \varepsilon) \geq m$, but then we have shown that in fact $-\lambda_m(\mathcal{M}_j^\bullet) \leq -E - \varepsilon < -E$. This proves item II.

It is true that $Z_j^\bullet(\lambda_{j,1}^\bullet) \geq 0$. Suppose that $Z_j^\bullet(\lambda_{j,k-1}^\bullet) \geq k-2$. But it follows from item 5 of Lemma B.2 that $Z_j^\bullet(\lambda_{j,k}^\bullet) \geq Z_j^\bullet(\lambda_{j,k-1}^\bullet) + 1 \geq k - 1$. On the other hand, notice that if $Z_j^\bullet(\lambda_{j,k}^\bullet) > k - 1$, then by item II, $-\lambda_{j,k} < -\lambda_{j,k}$. So we must have $Z_j^\bullet(-\lambda_{j,k}^\bullet) \leq k - 1$. This proves item III.

□

Now we can prove (4.9). We know from item II of Lemma B.3 that $N_E(\mathcal{M}_j^\bullet) \geq Z_j^\bullet(E)$. Since $Z_j^\bullet(E) < \infty$ if $-E < 0$, suppose that $N_E(\mathcal{M}_j^\bullet) > Z_j^\bullet(E) = m$, then by definition this implies that $-\lambda_{j,m+1}^\bullet \leq -E$, but then item 4 of Lemma B.2 and item III of Lemma B.3 imply that

$$Z_j^\bullet(E) \geq Z_j^\bullet(\lambda_{j,m+1}^\bullet) + 1 = m + 1.$$

This proves (4.9).

References

[1] S. Agmon. *Spectral theory of Schrödinger operators on Euclidean and on non-Euclidean spaces.* Comm. Pure Appl. Math., 39(S, suppl.):S3–S16, 1986. Frontiers of the mathematical sciences: 1985 (New York, 1985).

[2] D. Borthwick. *Spectral theory – basic concepts and applications.* Grad. Texts in Math., 284 Springer, Cham, [2020], ISBN:978-3-030-38001-4, ISBN:978-3-030-38002-1

[3] J-M. Bouclet. *Absence of eigenvalue at the bottom of the continuous spectrum on asymptotically hyperbolic manifolds.* Ann. Global Anal. Geom. 44 (2013), no. 2, 115–136.

[4] E. A. Coddington and N. Levinson. *Theory of ordinary differential equations.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[5] H. Donnelly and P. Li. *Pure point spectrum and negative curvature for noncompact manifolds.* Duke Math. J. 46, 497–503 (1979).

[6] E. B. Davies. *Spectral theory and differential operators.* Cambridge University Press, 1995.

[7] N. Dunford and J.T.Schwartz. *Linear operators.* Interscience, New York, 1958.

[8] G.M.L. Gladwell. *Inverse problems in scattering : an introduction.* Springer Science+Business Media Dordrecht (1993) DOI 10.1007/978-94-011-2046-3.

[9] C.R. Graham. *Volume and area renormalizations for conformally compact Einstein metrics.* Rend. Circ. Mat. Palermo (2) Suppl. No. 63 (2000), 31-42.

[10] A. Hassell and S. Marshall. *Eigenvalues of Schrödinger operators with potential asymptotically homogeneous of degree* -2. Trans. Amer. Math. Soc. 360 (2008), no. 8, 4145 –4167.

[11] P. Hislop and I. Sigal. *Introduction to spectral theory with applications to Schrödinger operators.* Applied Mathematical Sciences 113, Springer.

[12] L. Hörmander. *The Analysis of the linear partial differential operators, Vol. III.* Springer – Verlag, (1985).

[13] M. Joshi and A. Sá Barreto. *Inverse scattering on asymptotically hyperbolic manifolds.* Acta. Math. (2000) 184:41–86.

[14] W. Kirsch and B. Simon. *Corrections to the classical behavior of the number of bound states of Schrödinger operators.* Ann. Physics 183 (1988), no. 1, 122–130.

[15] R. Mazzeo. *The Hodge cohomology of a conformally compact metric.* J. Differential Geom. 28 (1988), no. 2, 309-339.

[16] R. Mazzeo. *Unique continuation at infinity and embedded eigenvalues for asymptotically hyperbolic manifolds.* Amer. J. Math. 113 (1991), 25–45.

[17] R. Mazzeo and R. Melrose. *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature.* J. Funct. Anal. 75 (1987), no. 2, 260-310.

[18] R. Mazzeo and R. McOwen. *Singular Sturm-Liouville theory on manifolds.* J. Differential Equations176(2001), no.2, 387–444.

[19] R. Melrose. *Geometric Scattering Theory.* Cambridge University Press, 1995.

[20] F. Olver. *Asymptotics and special functions.* Academic press, 2014.

[21] P. Perry. *The Laplace operator on a hyperbolic manifold. I. Spectral and scattering theory.* J. Funct. Anal., 75:161–187, (1987).

[22] P. Perry. *The Laplace operator on a hyperbolic manifold. II. Eisenstein series and the scattering matrix.* J. Reine. Angew. Math., 398:67?91, (1989).

[23] M. Reed and B. Simon *Methods of modern mathematical physics. I. Functional Analysis.* Academic Press, New York-London, 1978.

[24] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press, New York-London, 1978.

[25] A. Sá Barreto and M. Zworski. *Distribution of Resonances for Spherical Black Holes.* Mathematical Research Letters, 4; 103-121, 1997.

[26] W. Stenger. *On the Variational principles for eigenvalues for a class of unbounded operators.* Journal of Mathematics and Mechanics, 1968, Vol. 17, No. 7 (1968), pp. 641–648

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