Something Real Fun:

Positivity in Real Algebraic Geometry and Moment Problems in Functional Analysis

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Abstract

Expository notes for the Emory Graduate Student Algebra Seminar (RANT). We discuss the fundamental objects used in the study of positivity in Real Algebraic Geometry, their connections with measure theory and the moment problem, and introduce some applications to optimization.

1 Positivity in Real AG

Classical Geometry over $\mathbb{C}$: In classical algebraic geometry, we look at the zero sets of polynomials over an algebraically closed field such as $\mathbb{C}$. In particular, if we want to look at the zeros set of $S = \{g_1, \dots, g_m\} \subseteq \mathbb{C}[X_1, \dots, X_n] =: \mathbb{C}[\underline{X}]$. We then associate an algebraic object to the set S, the ideal

$$I = (g_1, \dots, g_m) = \{f = h_1 g_1 + \dots + h_m g_m | h_i \in \mathbb{C}[\underline{X}]\} \subseteq \mathbb{C}[\underline{X}],$$

which consists of all polynomials which are "obviously zero" when all of the g_i are zero. To this ideal, we can then associate a geometric object to the ideal, the variety

$$\mathcal{V}(I) = \{x \in \mathbb{C}^n | f(x) = 0 \, \forall f \in I\}.$$

We can then translate back to algebra by considering the ideal of polynomials which vanish on all of $\mathcal{V}(I)$,

$$\mathcal{I}(\mathcal{V}(I)) = \{f \in \mathbb{C}[\underline{X}] | f(x) = 0 \, \forall x \in \mathcal{V}(I)\},$$

the radical of the ideal I. In general the ideals I and $\mathcal{I}(\mathcal{V}(I))$ are different. A result about the relationship between $\mathcal{I}(\mathcal{V}(I))$ and I is called a Nullstellensatz.

Real Geometry over $\mathbb{R}$: When we switch to polynomials over $\mathbb{R}$, we lose the algebraic closedness of our base field, but gain a notion of positivity which is compatible with our field operations. This allows us to consider the nonnegativity set of $S = \{g_1, \dots, g_m\} \subseteq \mathbb{R}[\underline{X}]$.

To do so, we want to make an analogy with teh construction of the ideal generated by S. However, instead of taking the collection of polynomials that are "obviously zero" when the g_i are all zero, we take the collection of polynomials that are nonnegative when the g_i are. This gives the Preorder generated by S:

$$T = \{f = h_0 + \sum_{\epsilon \in \{0,1\}^m} h_\epsilon \mathbf{g}^\epsilon | h_i \in \sum \mathbb{R}[\underline{X}]^2\}$$

T is then the smallest subset of $\mathbb{R}[\underline{X}]$ such that $T + T \subseteq T$, $TT \subseteq T$, and $a^2 \in T$ for all $a \in \mathbb{R}[\underline{X}]$.

We can then associate to T a geometric object called the Semialgebraic set

$$K(T) = \{x \in \mathbb{R}^n | f(x) \geq 0 \, \forall f \in T\}.$$

This translates back into algebra by considering the Saturated Preorder

$$\mathcal{P}(K(T)) = \{f \in \mathbb{R}[\underline{X}] | f(x) \geq 0 \, \forall x \in K(T)\}.$$

As in the classical case, where $\mathcal{I}(\mathcal{V}(I)) \neq I$ in general, we have that $\mathcal{P}(K(T)) \neq T$. A result about the relationship between $\mathcal{P}(K(T))$ and T is called a Positivstellensatz or Nichtnegativstellensatz.

The relationships between Classical and Real objects are summarized in Table 1.

Classical ($\mathbb{C}[\underline{X}]$)	Real ($\mathbb{R}[\underline{X}]$)
Ideal $I = \{f = h_1 g_1 + \ldots + h_m g_m \| h_i \in \mathbb{C}[\underline{X}]\}$	Preorder $T = \{f = h_0 + \sum_{\epsilon \in \{\mathbf{0},\mathbf{1}\}^{\mathbf{m}}} h_\epsilon \mathbf{g}^\epsilon \| h_i \in \sum \mathbb{R}[\underline{X}]^2\}$
Variety $\mathcal{V}(I) = \{x \in \mathbb{C}^n \| f(x) = 0 \, \forall f \in I\}$	Semialgebraic Set $K(T) = \{x \in \mathbb{R}^n \| f(x) \geq 0 \, \forall f \in T\}$
Radical Ideal $\mathcal{I}(\mathcal{V}(I)) = \{f \in \mathbb{R}[\underline{X}] \| f(x) = 0 \, \forall x \in \mathcal{V}(I)\}$	Saturated Preorder $\mathcal{P}(K(T)) = \{f \in \mathbb{R}[\underline{X}] \| f(x) \geq 0 \, \forall x \in K(T)\}$.

Table 1: Summary of Classical vs Real objects

One classical Positivstellensatz is the following.

Theorem 1.1 (Krivine-Stengle Positivstellensatz)**.** *Let $S = \{g_1, g_2, \ldots, g_s\} \subseteq \mathbb{R}[\underline{X}]$, $K = \{g_i(x) \geq 0\}$ and T the preordering generated by S. Then,*

(i) $f > 0$ *on* $K \iff pf = 1 + q$ *for* $p, q \in T$

(ii) $f \geq 0$ *on* $K \iff pf = f^{2m} + q$ *for some* $m \geq 0$

(iii) $f = 0$ *on* $K \iff -f^{2m} \in T$ *for some* $m \geq 0$

(iv) $K = \emptyset \iff -1 \in T$.

Note that positivity and nonnegativity certificates in the Positivstellensatz require "denominators". We would like to find simpler certificates of positivity in special cases. One such case is when K is compact (in the Euclidean topology on $\mathbb{R}^n$).

2 Connections to Moment Problems

A convex cone C in a real vector space V is a convex set $C \subseteq V$ such that for any $x, y \in C$ and $\lambda, \mu \geq 0$, $\lambda x + \mu y \in C$. The dual cone to a convex cone C is defined as

$$C^* = \{\ell \in V^* | \, \ell(x) \geq 0 \text{ for all } x \in C\}.$$

Define the double dual of C as

$$C^{**} = \{v \in V | \ell(v) \geq 0 \text{ for all } \ell \in V^*\}$$

One important observation about a preorder T is that it forms a convex cone in the real vector space $\mathbb{R}[\underline{X}]$. For concreteness, suppose that K is a basic closed semialgebraic set. That is, fix $S = \{g_1, g_2, \ldots, g_m\} \subseteq \mathbb{R}[\underline{X}]$ and let $K = \{x \in \mathbb{R}^n | g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$. Then, the dual cone of $\mathcal{P}(K)$ can be described by the Riesz-Haviland Theorem:

Theorem 2.1 (Riesz-Haviland)**.** *A linear functional $\ell \in \mathbb{R}[\underline{X}]^*$ has $\ell \in \mathcal{P}(K)^*$ if and only if ℓ is integration with respect to a Borel measure on K.*

The Riesz-Haviland theorem also gives a partial answer to the moment problem:

Problem 2.1 (Moment Problem)**.** Given a linear functional $\ell : \mathbb{R}[\underline{X}] \to \mathbb{R}$, when is ℓ integration with respect to a Borel measure on K?

Theorem 2.1 says that we need to check that $\ell \in \mathcal{P}(K)^*$. We would like to reduce to cases where we can work with the preorder T generated by S, as T has the concrete description $T = \{f = h_0^2 + \sum_{\epsilon\in\{\mathbf{0,1}\}^{\mathbf{m}}} h_\epsilon^2 \mathbf{g}^\epsilon | h_i \in \mathbb{R}[\underline{X}]\}$. Proposition 2.2 below provides criteria for such a reduction. If any of the equivalent conditions hold, we say that T has the Strong Moment Property (SMP).

Proposition 2.2. *The following are equivalent:*

(i) $T^* = \mathcal{P}(K)^*$

(ii) $T^{**} = \mathcal{P}(K)$

(iii) For each $\ell \in T^*$*, there is a Borel measure* μ *on* K *such that* $\ell(f) = \int_K f d\mu$*.*

In particular, if T satisfies (SMP), then $\ell(f) \geq 0$ for all $f \in T$ implies that ℓ is integration with respect to a Borel measure. Schmüdgen's Theorem 2.3 gives a particularly useful case where the preordering T satisfies (SMP).

Theorem 2.3 (Schmüdgen)**.** *If* $f \in \mathbb{R}[\underline{X}]$ *is bounded on* K*, say* $|f| < k$*, then* $k^2 - f^2 \in T^{**}$*.*

Corollary 2.3.1 (Schmüdgen's Positivstellensatz)**.** *If* K *is compact and* $f > 0$ *on* K*, then* $f \in T$*.*

Schmüdgen's proof of Theorem 2.3 relies on the Krivine-Stengle Positivstellensatz as well as functional analytic tools.

3 Applications to Optimization

In this section, we briefly discuss the connections between certificates of positivity and optimization.

Polynomial Optimization The global polynomial optimization problem provides a good starting point for studying the relationship between positivity certificates and optimization:

Problem 3.1. [Global Polynomial Optimization] Given a polynomial $f \in \mathbb{R}[\underline{X}]$, compute

$$f^* = \inf_{x\in\mathbb{R}^n} f(x)$$

This problem is hard from an optimization point of view, as it is in general nonconvex. However, we can make the following observations to write an approximation to the global optimization problem

(i) $\gamma \in \mathbb{R}$ is a global lower bound for f if and only if $f - \gamma$ is globally nonnegative

(ii) A sufficient condition for global nonnegativity of a polynomial h is a representation of h as a sum of squares. Moreover, h is a sum of squares if and only if there is a symmetric positive semidefinite matrix Q such that $h = \nu(\underline{X})^T Q \nu(\underline{X})$, where $\nu(\underline{X})$ is a vector of monomials of degree at most $\frac{1}{2}\deg h$.

This motivates the following SOS relaxation:

Problem 3.2 (SOS Relaxation of 3.1)**.**

$$\begin{aligned} f^*_{SOS} = \max_{\gamma,Q} \quad & \gamma \\ \text{s.t.} \quad & f(\underline{X}) - \gamma = \nu(\underline{X}) Q \nu(\underline{X}) \\ & Q \succeq 0 \end{aligned}$$

This SOS relaxation has the form of a Semidefinite Program and can be solved numerically.

Generalized Moment Problem: We can similarly simplify more complicated optimzation problems. Consider the Generalized Moment Problem:

Problem 3.3 (Generalized Moment Problem (Primal))**.** Given a finite set Γ a subset $\Gamma_+ \subseteq \Gamma$, polynomials $h_j, g_i, f \in \mathbb{R}[\underline{X}]$, the semialgebraic set $K = \{x \in \mathbb{R}^n | g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$, and $\mathcal{M}(K)$ the set of finite measures on K:

$$
\begin{aligned}
f^*_{mom} = \sup_{\mu \in \mathcal{M}(K)_+} & \int_K f \, d\mu \\
\text{s.t.} & \int_K h_j \, d\mu \leq \gamma_j \qquad j \in \Gamma_+ \\
& \int_K h_j \, d\mu = \gamma_j \quad j \in \Gamma \setminus \Gamma_+
\end{aligned}
$$

The dual problem is then constrained by polynomial nonnegativity on K:

Problem 3.4 (Generalized Moment Problem (Dual))**.**

$$
\begin{aligned}
f^*_{pop} = \inf_{\lambda} & \sum_j \lambda_j \gamma_j \\
\text{s.t.} & \left(\sum_j \lambda_j h_j(x) \right) - f(x) \geq 0 \qquad x \in K \\
& \lambda_j \geq 0 \qquad\qquad\qquad\qquad j \in \Gamma_+.
\end{aligned}
$$

Using observation (ii) above as well as the fact that a sufficient condition for nonnegativity of a polynomial p on K is that p is an element of the Quadratic Module generated by $\{g_1, \ldots, g_m\}$[1], we can construct a hierarchy of strengthenings of 3.4:

Problem 3.5. For degree bounds d_i,

$$
\begin{aligned}
f^*_i = \inf_{\lambda, \sigma_\ell} & \sum_{j \in \Gamma} \lambda_j \gamma_j \\
\text{s.t.} & \sum_{j \in \Gamma} \lambda_j h_j - f = \sigma_0 + \sum_{\ell=1}^{m} \sigma_\ell g_\ell \\
& \sigma_\ell \in \sum \mathbb{R}[\underline{X}]^2 \text{ and } \deg \sigma_\ell g_\ell \leq d_i \\
& \lambda_j \geq 0, \qquad j \in \Gamma_+
\end{aligned}
$$

For each i, problem 3.5 is a semidefinite programming problem! Moreover, under some assumptions, the f^*_i will converge to the optimal solution of the primal generalized moment problem.

In each of the above problems, positivity results in real algebraic geometry help to develop computationally tractable methods for complicated optimization.

[1]that is, $p = \sigma_0 + \sum_{i=1}^m \sigma_i g_i$, where $\sigma_i \in \sum \mathbb{R}[\underline{X}]^2$

References: These notes were developed from reading out of the following reference books:

- Marshall, *Positive Polynomials and Sums of Squares*
- Blekherman, Parillo, and Thomas, *Semidefinite Optimization and Convex Algebraic Geometry*
- Lassere, *Moments, Positive Polynomials and Their Applications*