EMORY MATHEMATICS DIRECTED READING PROGRAM

Program description

The Emory Math Directed Reading Program (DRP) is a graduate student-run program aiming to pair undergraduate students with graduate student mentors to read and learn material that is not typically offered in a traditional course setting. Undergraduate students are expected to work mostly independently to read the text and attempt exercises, then meet regularly with their graduate student mentor to discuss the material.

Steering Committee

If you have any questions or concerns, or suggestions for future DRP topics, please reach out to Ylli Andoni at ylli.andoni@emory.edu, Guangqiu Liang at guangqiu.liang@emory.edu, Mitchell Scott at mitchell.scott@emory.edu, or Akash Sureshkumar at akash.sureshkumar@emory.edu.

Past Topics and Descriptions

Students specify a topic(s) of interest when applying to the program, in order to be matched with an appropriate graduate mentor. Selected topics divided by focus area are listed below. If a topic sounds interesting, please see the next page for more detailed sample descriptions.

Algebra.

- Category theory
- Commutative algebra
- Computational algebra
- Representation theory of finite groups

Analysis and Geometry.

- Analysis on manifolds
- Convex geometry
- Fourier analysis
- Geometry from Axioms
- Mathematical Physics
- Riemannian Geometry

Applied and Computational Math.

- Iterative Methods for Deblurring Images
- PDE Constrained Optimization
- Stochastic Optimization
- Stochastic Processes

Discrete Math.

- Graph Coloring
- Extremal Graph Theory
- Theoretical Computer Science

Logic.

- Model Theory
- Set theory

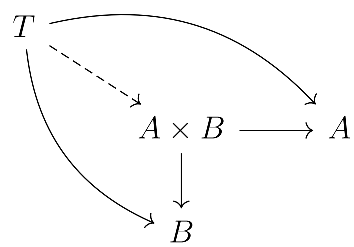
Number Theory.

- Algebraic number theory
- Computational number theory
- Elementary number theory
- Elliptic curves
- p-adic numbers

Course name: Category theory
Text: Chapter 1 of *Foundations of Algebraic Geometry*, by Ravi Vakil, and *Categories for the Working Mathematician*, by Saunders MacLane.
Prerequisites: Abstract algebra I/II (Math 421,422).

Description: How many different kinds of "products" have you seen? In linear algebra, we take the *Cartesian product* of vector spaces, e.g. $\mathbb{R}^2 \times \mathbb{R}^3$, but we can also take the *direct product* of groups, say $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. You may also have run into products of topological spaces, graphs, or product measures. What do these constructions have in common, and why are they deserving of their designation as a *product*?

Category theory suggests that instead of studying mathetmatical objects like vector spaces, groups, and topological spaces, we study the *maps between them*. For our product example above, the categorical perspective offers the following universal definition: given objects A and B, the product $A \times B$ is the (essentially) unique object denoted $A \times B$ such that *any other* object T with maps to A and B has a map to $A \times B$.



In this course we will explore ideas like this, learning about categories, functors, universal constructions, and more. While this abstract subject is rich and fascinating in its own right, we will use plenty of examples, focusing on how a categorical perspective can help to organize and shape our view of the mathematical world.

Course name: Commutative algebra
Text: *Introduction to Commutative Algebra*, by Michael Atiyah and Ian Macdonald.
Prerequisites: Abstract Algebra I (Math 421) or some familiarity with groups and rings would help, but isn't necessarily required.

Description: The goal of this course is to read through and work out the details in the first few chapters of Atiyah and Macdonald's *Introduction to Commutative algebra*. Each week, we will select a few theorems and exercises to write out in detail. We will focus on concrete examples and problem solving. Depending on time and interest, we can stop after the second chapter to apply what we have learned to geometric problems from Fulton's *Algebraic curves* and applications of the Nullstellensatz or continue with more commutative algebra. If that is the case, the goal is to define affine scheme. We will start by defining the Zariski topology on Spec$A$, verifying it is a topology, and finding a basis of open sets. Then we will define presheaf and sheaf and go over examples, with emphasis on the structure sheaf as described in Hartshorne. Finally, if time permits, we will define locally ringed space and affine scheme. As this is meant to be introductory, a student of any level is welcome (though if there is no familiarity with algebra then more meetings may be necessary).

Course name: Computational algebra
Text: *Ideals, Varieties, and Algorithms: an Introduction to Computational Algebraic Geometry and Commutative Algebra*, by David Cox, John Little, and Donald O'Shea.
Prerequisites: Linear algebra (Math 221 or equivalent). Some familiarity with rings (e.g. Math 421 and/or 422) and/or programming (e.g. Math 315) would be beneficial, but not necessary.

Description: Linear algebra has taken a central role in modern mathematics due in large part to the algebraic, geometric, and computational structures inherent in linear systems of equations. When we extend our focus from linear to polynomial systems of equations, the relevant algebraic, geometric, and computational tools become more involved. In this course, we will take a hands-on approach to develop the algebra of systems of polynomial equations as well as the geometry of their solution sets. Additionally, we will see how Gaussian Elimination from linear algebra leads to a notion of *Gröbner bases* and computational methods in algebraic geometry, such as Buchberger's algorithm and elimination theory. Throughout, we will emphasize examples which showcase how algebra, geometry, and computation are intertwined. Material can be adjusted according to student background and interest.

Course name: Representation theory of finite groups
Text: *Representation Theory: a First Course*, by William Fulton and Joe Harris. Specifically, Part I: Chapters 1 – 5.
Prerequisites: Linear algebra (Math 221 or equivalent) and Abstract algebra I (Math 421).

Description: Representation theory aims to answer the question: given a group G, how can we describe all the ways in which G may be embedded in (or mapped to) a linear group $\mathrm{GL}(V)$? In this course, we will study representation theory of finite groups. This is in an effort to understand the main motivations and goals of representation theory in a setting where the nature of the groups G is relatively simple. We will spend the majority of our time exploring representations of familiar groups such as the symmetric and alternating groups. This will set the groundwork for any future study of finite-dimensional representations of Lie groups and Lie algebras.

Course name: Analysis on manifolds
Text: *Analysis on Manifolds*, by Marcello Seri.
Prerequisites: Linear algebra (Math 221), Real analysis I (Math 411). Real analysis II (Math 412) might be helpful but is not necessary.

Description: Manifolds arise naturally in many areas of mathematics, mostly in geometry and mathematical physics. They help us to generalize results from real analysis to more abstract mathematical objects. We can equip manifolds with different structures, and depending on the structure, calculus, Riemannian geometry, classical mechanics, Hamiltonian mechanics, general relativity and many different subjects can be studied.

We will be interested in the structure that allows us to do calculus and analysis on these manifolds. During this course we will generalize the concepts of differentiation to smooth manifolds, study vector fields on manifolds and their relation to dynamical systems and flows. We will also study integration along curves, and depending on time and student interest, Lie groups and Lie algebras or tensor fields.

Since the theory of manifolds is a very broad subject, there are a lot of books which we could also use during this course, for example Vector Calculus by Klaus Jänich and Manifolds, Tensor Analysis, and Applications by Ralph Abraham, Jerrold E. Marsden, Tudor Ratiu.

Course name: Convex geometry
Text: *A Course in Convexity*, by Alexander Barvinok.
Prerequisites: Linear algebra (Math 221 or equivalent), Real analysis I (Math 411) or familiarity with metric spaces.

Description: A subset C of a real vector space is convex if for any two points $x, y \in C$, the line segment connecting x and y is contained in C. For example, a pancake is convex while a donut is nonconvex. Convex sets are highly structured and appear across many areas of pure and applied math, including analysis, algebraic geometry, and optimization. The goals of this reading course will be to understand basic properties and structure of convex sets, duality theory in convex geometry, and applications to other areas of mathematics. Due to the prevalence of convexity in mathematics, applications can be highly tailored to student experience and interest.

Course name: Fourier analysis
Text: *Fourier Analysis: An Introduction*, by Stein and Shakarchi.
Prerequisites: Integral and multivariable calculus (Math 112, 211). Real analysis I (Math 411) would be helpful but not required.

Description: How does a string vibrate when you pluck it? How does heat diffuse when you turn on the stove? What is the largest area a rope of fixed length can enclose? How badly can a continuous function fail to be differentiable? Can one see an internal organ from external measurements? The study of Fourier transform and Fourier series manifests itself in various fields of mathematics such as real analysis, partial differential equation, number theory, and inverse problems.

In this reading project, we will look into basic elements of Fourier analysis and its wide applications in solving the wave equation and heat equation, obtaining the isoperimetric inequality, inverting the Radon transform, and producing an elegant proof of the Dirichlet's theorem on arithmetic progressions. Except the prerequisites, all basic materials will be developed from the scratch with various applications given depending on participants' interests.

Course name: Geometry from Axioms
Text: *Euclidean and Non-Euclidean Geometries*, by Charles A. Coppin.
Prerequisites: Foundations of Mathematics (Math 250)

Description: In 1899 Hilbert introduced a set of axioms to replace those set about by Euclid in 300 BC. This was a shift in mathematical thinking towards proof-based axiomatic systems, or the modern mathematics we do today. These axioms, either by Hilbert or earlier by Euclid, define Euclidean space, or real space as we know it, and while the end statements are intuitive, the logic needed to prove them is precise and rewarding. We will define, precisely and from simple set theory, what a point in space is, what a line is, what it means to be between two points, what collinearity is, what convexity is, what a triangle is, what an angle is, and work out theorems about these objects.

The goal of this course is to understand this axiomatic way of thinking, to practice precise proof writing, and understand Euclidean Geometry. This topic can be extended to explorations in either other geometric settings, like the hyperbolic or spherical geometry, or to interesting objects embedded in Euclidean space.

Course name: Mathematical Physics
Text: *Mathematical Methods of Classical Mechanics*, by Vladimir Arnold.
Prerequisites: Vector Calculus (Math 211), and some exposure to Mathematical analysis such as Real Analysis I (Math 411) would be helpful but not required.

Description: Many different mathematical methods and concepts are used in classical mechanics such as differential equations, flows and smooth manifolds. The study of flows enables us to derive some qualitative properties of a given mechanical system. Moreover, manifolds arise as natural objects in the study of Holonomic mechanics.

Course name: Riemannian Geometry
Text: *Introduction to Riemannian Manifolds*, by John M. Lee
Prerequisites: Vector Calculus (Math 211), Real Analysis I (Math 411) and Topology/ Differential geometry (Math 344) would be helpful but not required.

Description: Riemannian manifolds are manifolds endowed with Riemannian metrics, which are essentially rules for measuring lengths and angles between tangent vectors. It is the most "geometric" branch of differential geometry. In this project, we plan to take a sightseeing approach to Riemannian geometry by examining the interplay among fundamental objects such as geodesics, curvature and connection. If time permits, we will also explore applications of Riemannian geometry in the study of General Relativity.

Course name: Iterative Methods for Deblurring Images
Text: *Iterative Methods and Preconditioning for Large and Sparse Linear Systems with Applications*, by Daniele Bertaccini, Fabio Durastante.
Prerequisites: Numerical Analysis (Math 315)

Description: Iterative methods are the workhorses of the computational mathematics world, especially when it comes to linear systems $Ax = b$, where A is a matrix, x is the solution, and b is the output. In some applications like image deblurring, naively inverting the blurring matrix A, is a bad idea as it is an "ill-conditioned problem", so instead we employ iterative methods that get us closer and closer to the true solution without losing the information we care about. In this course, we will start by studying stationary iterative methods like Jacobi, Gauss-Seidel, and Successive Over Relaxation. We will then turn our attention to the broader class of Krylov subspace methods, which are the gold-standard. We hope to end the course by discussing preconditioners for these methods to make them converge to the true solution faster.

Course name: PDE Constrained Optimization
Text: *Perspectives in Flow Control and Optimization*, by Max Gunzburger.
Prerequisites: Linear Algebra (Math 221). Differential Equations (Math 211).

Description: Many physical systems can be described using partial differential equations. To perform optimization in these systems, we need to ensure that the PDE constraints set by the physical system are obeyed. The two variables of interest in these types of problems are state variables and control variables. In fluid mechanics, the state variable might be the velocity of the fluid at a particular point, and the control variable might be the initial velocity at the inflow. The goal in this case would be to determine what inflow velocity would result in the desired velocity at the point of interest.

Transitioning from the finite to infinite dimensional setting introduces additional challenges in setting up and solving the optimization problem, but many of the tools from nonlinear optimization and numerical PDEs can still be used.

Course name: Stochastic Optimization
Text: *Reinforcement Learning and Stochastic Optimization: A Unified Framework for Sequential Decisions*, by Warren Powell.
Prerequisites: Multivariable Calculus (Math 211), Linear Algebra (Math 221)

Description: Various optimization methods have or intentionally incorporate randomness. There might be random uncertainty in the quantity we are optimizing, or we might intentionally inject randomness into our optimization method to improve it in some way. In this course, we will investigate when stochastic optimization methods are utilized and explore specific optimization algorithms which leverage stochasticity.

Course name: Stochastic Processes
Text: *Essentials of Stochastic Processes*, by Richard Durrett.
Prerequisites: Probability/ Statistics (Math 207 or Math 361/362), Linear Algebra (Math 221)

Description: Stochastic processes, or the evolution of a series of randomly happening events, are tremendously useful for modeling phenomena that happen in biology, physics, social networks, and many other fields. In this course, we start by studying random walk, Markov chains, and Poisson processes. We explore their properties such as memoryless-ness, return time, exit time, and independence, with the eventual goal of tackling martingales. The course will culminate in discussing stochastic processes occurring in mathematical finance like calls, puts, pricing models, and the famous Black-Scholes formula.

Course name: Graph Coloring
Text: *Introduction to Graph Theory* , by Douglas B. West, and Graph Coloring and the Probabilistic Method, by Michael Molloy, Bruce Reed
Prerequisites: Foundations of Mathematics (Math 250)

Description: You may have noticed that many world maps give a color to each territory to make them easier to distinguish. Of course, this only works if bordering territories always get distinct colors. We could always color the map so that every country gets its own private color, but this seems wasteful. What's the fewest number of colors that we need? Certainly, we could draw maps that would need at least four colors by making four territories which all border each other (like the Google Chrome logo). Are four colors always enough? It turns out that the answer is YES! This question perplexed mathematicians for centuries until finally in 1977, Appel and Haken gave a computer-assisted proof of this fact, known as the Four Color Theorem.

Maps aren't the only things that need to be colored while requiring that certain pairs of objects receive different colors. This is what we call graph coloring. How many time slots do we need in order for each of our committees to be able to meet? (Committees with common members obviously can't meet at the same time.) How many groups do we need to break our 3rd grade class into in order to separate all of the "problem students" from their friends? These kinds of problems are computationally difficult in general, but sometimes, if we know something about the structure of what we're dealing with (e.g., it comes from a map), the problem becomes feasible.

Course name: Extremal Graph Theory
Text: *Extremal Graph Theory*, by Béla Bollobas
Prerequisites: Graph theory (Math 487)

Description: We plan to explore basic extremal problems and properties of graphs such as cycles, colorings, diameter and girth. Lastly, we look into structured graphs such as complete graphs, matchings, trees, etc.

Course name: Theoretical Computer Science
Text: *Introduction to the Theory of Computing*, by Michael Sipser
Prerequisites: Foundations of Mathematics (Math 250), Exposure to logic or discrete math might be beneficial but is not required

Description: When we think of a computer, the biggest question we might ask is: Mac or PC? But did you know there are many levels of computation? In this DRP, we aim to explore these levels, starting with a Deterministic Finite Automaton, then going to Pushdown Automata, then going to Turing Machines. We learn skills to determine if these levels of computation can or cannot parse a specific language such as the Pumping Lemma. Lastly, we aim to show that even though Turing Machines are the strongest form of computation, there are some languages that we cannot know if it meets the criteria or not; these are called undecidable. If there is time permitting, we aim to dicuss certain complexities of the decidable languages, such as P vs. NP, and LogSpace vs PSpace.

Course name: Model Theory
Text: *A First Journey Through Logic* *Introduction to Graph Theory* , by Douglas B. West, and Graph Coloring and the Probabilistic Method, by Michael Molloy, Bruce Reed
Prerequisites: Foundations of Mathematics (Math 250)

Description: First we start with introductory model theory and first order logic. We delve into what are formal proofs and semantics. This is all a build-up to the most famous theorem in logics - Gödel's completeness theorem and first incompleteness theorem.

Course name: Set theory
Text: *Set Theory and Logic*, by Robert R. Stoll.
Prerequisites: Foundations of math (Math 250) or similar exposure to proofs.

Description: The theory of sets is home to some of the most philosophically challenging and counter-intuitive results in all of mathematics. The goal of this course is to explore set theory at a level deeper than is covered in Foundations of Mathematics, and our initial plan is to work through as much of Chapters 2, 3, 5, and 7 of Stoll's *Set Theory and Logic* as possible. Depending on interest and time, however, we might branch out into other readings and topics.

Course name: Algebraic number theory
Text: *TIFR pamphlets on algebraic number theory* and *Algebraic Theory of Numbers*, by Pierre Samuel.
Prerequisites: Abstract algebra I/II (Math 421,422), Abstract vector spaces (Math 321), and some commutative algebra.

Description: The idea of the course is to amalgamate one's interest in algebra with number theory. We will read through and work out the details of the TIFR pamphlets and move on to Samuel's book from there. The goal of this short course would be to build the basics necessary to concretely understand the meaning and applications of the "Lagrange's Four Squares Theorem," which states that any natural number can be represented as the sum of four integer squares. Depending upon time and interest, we'll try to go deeper into understanding the quadratic class number formula. The pacing of the course is flexible since my goal is to make learning this topic fun!

Course name: Computational number theory
Text: *Computational Number Theory*, by Abhijit Das.
Prerequisites: Foundations of mathematics (Math 250 or equivalent). Some familiarity with programming would be helpful but not required.

Description: The field of number theory usually deals with the study of integer and rational numbers. After the recent invention of public-key cryptography, number theory became an important area of study for computer scientists. In particular, the implementation of public-key encryption algorithms, such as RSA, are built from the foundations of computational number theory. In this course we will study a range of basic number-theoretic algorithms via the computer algebra system GP/PARI. Topics include implementation of algorithms such as the Euclidean GCD algorithm, modular exponentiation, Chinese remainder theorem, Hensel lifting, quadratic residues and non-Residues, Legendre symbol, Jacobi symbol and more.

Course name: Elementary number theory
Text: *A Friendly Introduction to Number Theory*, by Joseph Silverman.
Prerequisites: Foundations of Mathematics (Math 250).

Description: The idea of the course is to give number theory a chance! We will read through and work out the details of Silverman's book. The text has a lot of simple yet challenging exercises at the end of each chapter. A major part of this DRP would be to solve these problems and come up with your own conjectures! The goal of this short course would be to build the basics necessary to concretely understand the meaning and applications of "Lagrange's Four Squares Theorem," which states that any natural number can be represented as the sum of four integer squares. The pacing of the course is flexible since my goal is to make learning this topic fun!

Course name: Elliptic curves
Text: *Rational Points on Elliptic Curves*, by Joseph Silverman and John Tate.
Prerequisites: Abstract algebra I, preferably also II (Math 421, 422). Number theory (Math 328) would help, but isn't required.

Description: Elliptic curves have played a central role in number theory for over a century. While they arose from classical complex geometry, they have more recently been used in cryptographic applications, e.g. keeping online credit card transactions secure. The subject also provides a wonderful entry point to the world of arithmetic and algebraic geometry and the study of rational points on curves and abelian varieties. Some familiarity with groups and finite fields (Abstract Algebra I and possibly II) is necessary, and any additional number theoretic or analytic background will be useful.

The first goal of this course is to understand that the points on an elliptic curve have the structure of an abelian group. Indeed, this group is finitely generated, a fact known as the Mordell–Weil theorem. To understand the proof, we begin by investigating points of finite order (a.k.a. torsion points), then we will build a theory of heights to complete the proof of the theorem. We will also spend some time thinking about elliptic curves over finite fields — the setting of their cryptographic applications. This could lead to working through toy examples of elliptic curve cryptography and/or factoring algorithms, which may especcially interest those with programming experience or an interest in learning!

Course name: p-adic numbers
Text: *p-adic numbers: An introduction*, by F. Gouvea.
Prerequisites: Abstract algebra I/II (Math 421/422), Real analysis I (Math 411). Number theory (Math 328) would help, but isn't required.

Description: In this course, we will study p-adic numbers, which play a central role in modern number theory, laying the foundation for other topics such as class field theory and arithmetic geometry. The p-adic numbers arise from solving integer congruences, and come with interesting algebraic and analytic structures which make them useful to so many mathematicians. We will spend some time constructing and exploring the p-adics, building to key ideas such as Hensel's lemma and local-global principles. Time permitting, we may investigate further topics such as p-adic analysis and Newton polygons.

The textbook, Gouvea's *p-adic numbers: An introduction*, is a fun and conversational text with exercises sprinkled throughout. It should be an enjoyable read for students with a wide range of experience. The key prerequisite topics are groups, rings, fields, and congruences from algebra, and the basic ideas of metric topology and convergence, which should be covered in an introductory real analysis course. Experience with number theory may be helpful for motivation.