Characterizing Graphs Achieving an Upper Bound on Chromatic Index Involving Girth (Sttefen's Bound)

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The chromatic index $\chi'(G)$ of a graph G satisfies Vizing's classical bound $\chi'(G) \leq \Delta(G) + \mu(G)$, where $\Delta(G)$ and $\mu(G)$ denote the maximum degree and multiplicity, respectively. Steffen refined this bound by incorporating the girth $g(G)$, establishing that $\chi'(G) \leq \Delta(G) + \lceil \mu(G)/\lfloor g(G)/2 \rfloor \rceil$. Ring graphs (obtained by duplicating edges of odd cycles) are known to achieve this bound with equality. Stiebitz et al. posed two fundamental questions: Which parameter triples (Δ, μ, g) achieve Steffen's bound? Are there non-ring graphs achieving the bound?

We provide a negative answer to the second question for graphs with girth $g \geq 5$ and $\chi'(G) \geq \Delta + 2$. Specifically, we prove that if G is a critical graph achieving Steffen's bound with $g(G) \geq 5$ and $\mu(G) \geq \lfloor g(G)/2 \rfloor + 1$, then G must be a ring graph of odd girth. Our proof employs structural decomposition techniques based on cycle partitions, density arguments using the recently confirmed Goldberg-Seymour Conjecture, and careful analysis of neighborhood structures relative to shortest cycles. The case $g = 4$ remains open, presenting intriguing challenges for future research. This is a joint work with Guantao Chen, Jessica McDonald and Anna Yu.