Abelian Groups with 3-Chromatic Cayley Graphs

Maya Sankar

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Let G be an abelian group. We show that there exists a Cayley graph on G with chromatic number 3 if and only if G has exponent not equal to 1, 2, or 4. This extends a result due to Payan, who proved that a Cayley graph on $\mathbb{F}_2^n$ cannot have chromatic number 3. Our proof generalizes the $k = 1$ case of a pivotal result of Lovász, which states that a graph G has chromatic number at least $k + 3$ if the first k homotopy groups of an associated topological space $\mathcal{N}(G)$ are trivial. Specifically, we show that if $\mathcal{N}(G)$ is connected and $\pi_1(\mathcal{N}(G))$ is torsion, then G has chromatic number at least 4. The proof is completely combinatorial and no experience with algebraic topology will be assumed.

This result is joint with Mike Krebs.