Bi-Parametric Operator Preconditioning

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joint work with

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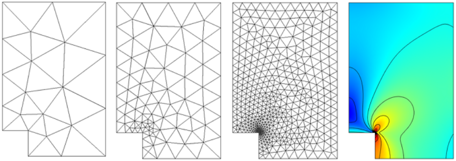
Outline

(Petrov)-Galerkin methods



1. **Continuous operator analysis:**
   X, Y reflexive Banach spaces, $a \in \mathcal{L}(X \times Y; \mathbb{C})$, $b \in Y'$.

$$\text{Seek} \quad u \in X \quad \text{such that} \quad a(u, v) = b(v), \quad \forall\ v \in Y.$$

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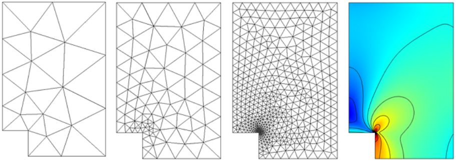

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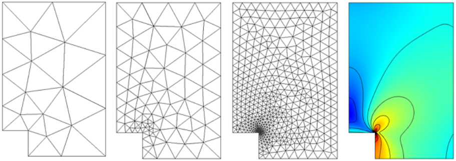

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3. **Linear system:** Pick bases in X_h and Y_h.

$$\text{Seek} \quad \mathbf{u} \in \mathbb{C}^N \quad \text{such that} \quad \mathbf{A}\mathbf{u} = \mathbf{b}.$$

Duality product

(Continuous dual pairing)

We identify $\mathrm{a} \in \mathcal{L}(X \times Y; \mathbb{C})$ and $\mathrm{A} \in \mathcal{L}(X; Y')$ through:

$$\langle \mathrm{A}u, v \rangle_{Y' \times Y} := a(u, v), \quad \forall\, u \in X,\ \forall\, v \in Y.$$

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Weak continuous problem:

$$\text{seek} \quad u \in X \quad \text{such that} \quad \mathrm{a}(u, v) = b(v), \quad \forall\, v \in Y$$

Strong continuous problem:

$$\text{seek} \quad u \in X \quad \text{such that} \quad \mathrm{A}u = b$$

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(Discrete dual pairing)

We identify $\mathsf{a} \in \mathcal{L}(X_h \times Y_h; \mathbb{C})$ and $\mathsf{A}_h \in \mathcal{L}(X_h; Y_h')$ through:

$$\langle \mathsf{A}_h u_h, v_h\rangle_{Y_h' \times Y_h} := a(u_h, v_h), \quad \forall\, u_h \in X_h,\ \forall\, v_h \in Y_h$$

Weak discrete problem:

$$\text{seek} \quad u_h \in X_h \quad \text{such that} \quad \mathsf{a}(u_h, v_h) = b(v_h), \quad \forall\, v_h \in Y_h.$$

Strong discrete problem:

$$\text{seek} \quad u_h \in X_h \quad \text{such that} \quad \mathsf{A}_h u_h = b_h.$$

Discrete inf-sup

(Discrete inf-sup)

Discrete inf-sup: There exists $\gamma_{\mathsf{A}} > 0$ such that:

$$\forall\ u_h \in X_h, \quad \gamma_{\mathsf{A}} \|u_h\|_X \leq \|\mathsf{A}_h u_h\|_{Y'}$$

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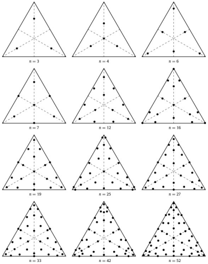
$\Rightarrow$ Discrete and matrix problems well-posed

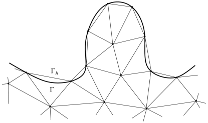
What about other approximation errors?

Approximations induced by:

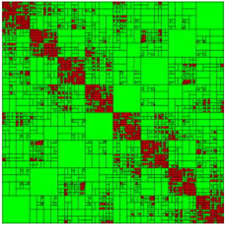
- Machine precision
- Quadrature rules
- Geometrical error (isogeometric analysis, curvilinear elements)
- "Fast methods" (FMM, H-mat)



Quadrature rules



Geometrical error



Hierarchical matrix

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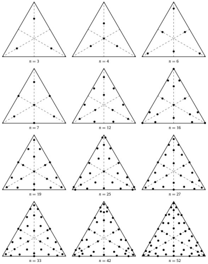

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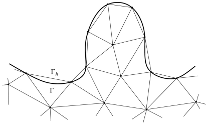

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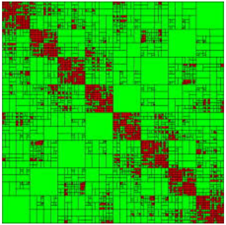

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⇒ Well-posedness? Controlled error?

First Strang's Lemma

Perturbed problem for $0 \leq \nu < 1$ such that:

$$\|\mathsf{A}_h - \mathsf{A}_{h,\nu}\|_{Y_h'} \leq \gamma_{\mathsf{A}}\nu \quad \text{and} \quad \|b_h - b_{h,\nu}\|_{Y_h'} \leq \nu \|b_h\|_{Y_h'}$$

$$\mathbf{A}_\nu \mathbf{u}_\nu = \mathbf{b}_\nu. \tag{1}$$

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Lemma (First Strang's Lemma[1])

Set $\nu \in [0,1)$ and let $u_{h,\nu} \in X_h$ and $u \in X$ be the unique solutions to the discrete perturbed and continuous problems, respectively. It holds that

$$\begin{aligned}\|u - u_{h,\nu}\|_X &\leq \inf_{w_h \in X_h} \left(\left(1 + \frac{\mathsf{K}_{\mathsf{A}}}{1-\nu}\right) \|u - w_h\|_X + \frac{\nu}{1-\nu}\|w_h\|_X \right) + \frac{\nu}{\gamma_{\mathsf{A}}(1-\nu)}\|b_h\|_{Y_h'} \\ &\leq (1 + \mathsf{K}_{\mathsf{A}}) \left(1 + \frac{\mathsf{K}_{\mathsf{A}}}{1-\nu}\right) \inf_{w_h \in X_h} \|u - w_h\|_X + \frac{2\nu}{\gamma_{\mathsf{A}}(1-\nu)}\|b_h\|_{Y_h'}\end{aligned}$$

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$$\leq (1 + \mathsf{K}_{\mathsf{A}}) \left(1 + \frac{\mathsf{K}_{\mathsf{A}}}{1-\nu}\right) \inf_{w_h \in X_h} \|u - w_h\|_X + \frac{2\nu}{\gamma_{\mathsf{A}}(1-\nu)} \|b_h\|_{Y_h'}$$

For small ν:

- Quasi-optimality constant $(1 + \mathsf{K}_{\mathsf{A}})^2$
- $\mathcal{O}(\nu)$-errors induced by A_ν and b_ν (e.g., $\nu = \mathcal{O}(h^{p+1})$)

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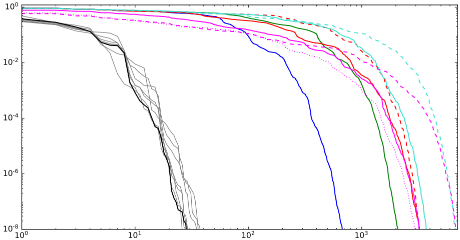
What about Iterative Solvers?

We solve $\mathbf{Au} = \mathbf{b}$.

✓ Set $\mathbf{u}_0$ and seek $(\mathbf{u}_k)_{k=1}^{N}$ such that $\mathbf{u}_k \to \mathbf{u}$

✓ Define $\mathbf{r}_k := \mathbf{Au}_k - \mathbf{b}$ for $0 \leq k \leq N$

- SPD or HPD matrix ⇒ Conjugate Gradient (e.g., Laplace: $-\Delta$)
- Indefinite matrix ⇒ GMRES or GMRES(m) (e.g., Helmholtz/Maxwell)



Relative l^2-residual norm error vs. iteration count k

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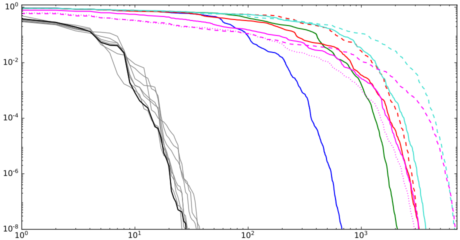

Relative l^2-residual norm error vs. iteration count k

Goal: $\frac{\|\mathbf{r}_k\|}{\|\mathbf{r}_0\|} \to 0$ as fast as possible (and h-independently)

CG: Condition number and spectral properties

- For HPD matrices, convergence for CG depends on the spectral condition number:

$$\kappa_S(\mathbf{A}) := \frac{|\lambda_{\max}(\mathbf{A})|}{|\lambda_{\min}(\mathbf{A})|}$$

- We apply CG to $\mathbf{A}\mathbf{u} = \mathbf{b}$, para $\mathbf{u}_0 = \mathbf{b} \Rightarrow (\mathbf{u}_k)_{k\geq 1}$.

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Lemma (Linear bound for CG)

The residual error of CG at iteration $1 \leq k \leq N$ *yields:*

$$\frac{\|\mathbf{u}_k - \mathbf{u}\|_A}{\|\mathbf{u}_0 - \mathbf{u}\|_A} \leq \left(1 - \frac{1}{\sqrt{\kappa_S(\mathbf{A})}}\right)^k =: \varrho^k$$

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- For Laplace $\kappa_S(\mathbf{A}) = \mathcal{O}(h^2)$
- h-dependence comes from a mismatch between spaces.
- When $h \to 0$: $\kappa_S(\mathbf{A}) \to \infty$, and $\varrho \to 1$.

Solution: Preconditioning (why we are here!)

We seek to solve

$$\mathbf{A}\mathbf{u} = \mathbf{b}.$$

Seek $\mathbf{P}$ such that:

1. $\mathbf{P}$ is relatively cheap to compute
2. $\mathbf{P}\mathbf{A} \approx \mathbf{I}$ or iterative solvers perform better than on the original system

$$\text{Seek} \quad \mathbf{u} \in \mathbb{C}^N \quad \text{such that} \quad \mathbf{P}\mathbf{A}\mathbf{u} = \mathbf{P}\mathbf{b}$$

Operator Preconditioning : PG Setting

(Continuous problem)

For X, Y reflexive Banach spaces, $\mathrm{A} \in \mathcal{L}(X; Y')$ with norm $\|\mathrm{a}\|$, $b \in Y'$:

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- We use the OP framework [Hiptmair, 2006] and introduce bounded linear operators $\mathrm{C}, \mathrm{N}, \mathrm{M}$ such that:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{A}} & Y' \\ {\scriptstyle \mathrm{M}^{-1}}\uparrow & & \downarrow{\scriptstyle \mathrm{N}^{-1}} \\ W' & \xleftarrow[\mathrm{C}]{} & V \end{array}$$

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- Set $\mathsf{P} := \mathsf{M}^{-1}\mathsf{C}\mathsf{N}^{-1}$
- There holds that $\mathsf{PA} \in \mathcal{L}(X; X)$
- One arrives at problem:

Problem (OP-PG)

$$\textit{Seek} \quad \mathbf{u} \in \mathbb{C}^N \quad \textit{such that} \quad \mathbf{PAu} = \mathbf{Pb}$$

- Bubnov-Galerkin $Y = X$, $V = W$, $\mathsf{N} = \mathsf{M}^*$
- Opposite-order Preconditioning $Y = X'$, $W = V'$, $\mathsf{N} = \mathsf{M} = I$

Theorem (Estimates for OP-PG[1])

Consider OP-PG. There holds that:

$$\kappa_S(\mathbf{PA}) \leq \frac{\|\mathsf{m}\|\|\mathsf{n}\|\|\mathsf{c}\|\|\mathsf{a}\|}{\gamma_{\mathsf{M}}\gamma_{\mathsf{N}}\gamma_{\mathsf{C}}\gamma_{\mathsf{A}}} =: \mathsf{K}_\star$$

Furthermore, the Euclidean condition number satisfies

$$\kappa_2(\mathbf{PA}) \leq \mathsf{K}_\star\, \mathsf{K}_{\Lambda_h}^2$$

with

$$\mathsf{K}_{\Lambda_h} := \frac{\sup_{u_h \in X_h \setminus \{\mathbf{0}\}} \frac{\|u_h\|_X}{\|\mathbf{u}\|_2}}{\inf_{u_h \in X_h \setminus \{\mathbf{0}\}} \frac{\|u_h\|_X}{\|\mathbf{u}\|_2}}$$

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New!

- ✓ Extension of OP for PG methods
- ✓ Bound for the Euclidean condition number

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Main ideas for efficient schemes

- Apply Operator Preconditioning[2]
- Use different tolerances for the preconditioner and stiffness matrix

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Iterative solvers performance is robust with respect to operators perturbations.

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Bi-Parametric Operator Preconditioning

For $\mu, \nu \in [0, 1)$, introduce suitably defined C_μ, A_ν, and b_ν.

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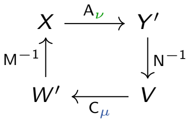

- There holds that $\mathsf{P}_\mu \mathsf{A}_\nu := (\mathsf{M}^{-1}\mathsf{C}_\mu \mathsf{N}^{-1})\mathsf{A}_\nu \in \mathcal{L}(X; X)$.
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Problem (OP-PG)

Seek $\mathbf{u}_\nu \in \mathbb{C}^N$ *such that* $\mathbf{P}_\mu \mathbf{A}_\nu \mathbf{u}_\nu = \mathbf{P}_\mu \mathbf{b}_\nu$

Theorem (Bi-Parametric Operator Preconditioning[1])

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$$\kappa_S(\mathbf{P}_\mu \mathbf{A}_\nu) \leq \mathsf{K}_\star \left(\frac{1+\mu}{1-\mu}\right)\left(\frac{1+\nu}{1-\nu}\right) =: \mathsf{K}_{\star,\mu,\nu}$$

and for the Euclidean version

$$\kappa_2(\mathbf{P}_\mu \mathbf{A}_\nu) \leq \mathsf{K}_{\star,\mu,\nu}\, \mathsf{K}^2_{\Lambda_h}$$

- ✓ Controlled condition numbers w.r.t. h, μ, ν
- ✓ No cross-terms in μ and ν
- ✓ Bounded $\mu, \nu \Rightarrow$ bounded $\kappa_S(\mathbf{P}_\mu \mathbf{A}_\nu)$

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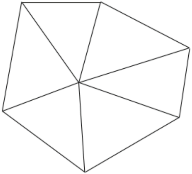
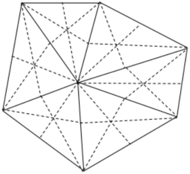
Application: Acoustic and EM time-harmonic wave scattering

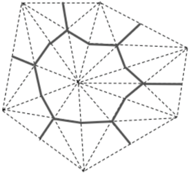
- Traditionally solved via boundary integral equations
- Operator preconditioner in the form of Calderón (opposite order) preconditioning
- Preconditioner built via dual mesh
- Dual mesh is achieved via barycentric refinement leading to expensive computational costs



Primal Mesh



Barycentric Mesh



Dual Mesh

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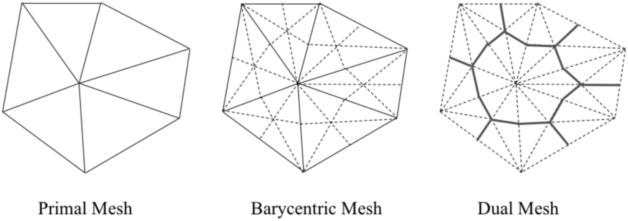
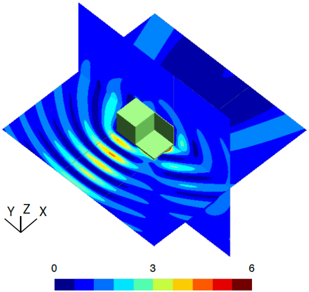
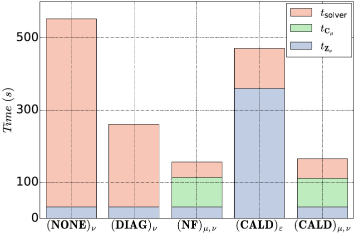
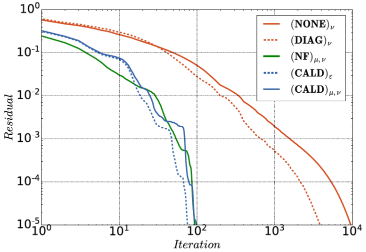


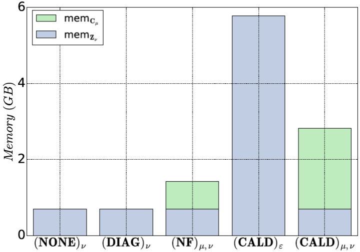
- Solution? Coarse approximation of **P**:
- Draback

EM Ficher cube with $k = 10$, $r = 10$, $N = 16,113$, $N_b = 96,678$, GMRES(200), $\nu = 1e-5$, $\mu = 1e-1$









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- Symmetric case: bounded $\kappa_S \Rightarrow$ Bounded linear convergence rate ✓
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- $X \equiv H$ with H Hilbert space with

$$\forall \; u_h, v_h \in X_h, \quad (u_h, v_h)_H = (\mathbf{Hu}, \mathbf{v})_2 =: (\mathbf{u}, \mathbf{v})_H$$

- Matrix H-FoV of $\mathbf{Q} \in \mathbb{C}^{N \times N}$

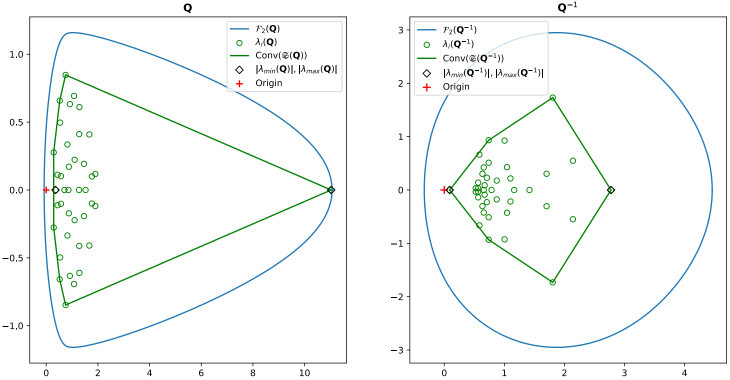
$$\mathcal{F}_H(\mathbf{Q}) := \left\{ \frac{(\mathbf{Qu}, \mathbf{u})_H}{(\mathbf{u}, \mathbf{u})_H} : \mathbf{u} \in \mathbb{C}^N \setminus \{\mathbf{0}\} \right\}$$

- Distance of $\mathcal{F}_H(\mathbf{Q})$ from the origin

$$\mathcal{V}_H(\mathbf{Q}) := \min_{z \in \mathcal{F}_H(\mathbf{Q})} |z| = \min_{\mathbf{u} \in \mathbb{C}^N \setminus \{\mathbf{0}\}} \frac{|(\mathbf{Qu}, \mathbf{u})_H|}{(\mathbf{u}, \mathbf{u})_H}$$

- Discrete H-FoV for Q_h and $\mathcal{V}_H(\mathsf{Q}_h)$ defined in the same fashion for any $\mathsf{Q}_h : X_h \to X_h$.

Example of 2-FoV



2-FoV boundary (blue line), eigenvalues (green circles), convex hull for eigenvalues (green line) and $|\lambda_{\min}|, |\lambda_{\max}|$ (black diamonds) for a matrix $\mathbf{Q} := \mathbf{I} + 0.5\mathbf{E} \in \mathbb{R}^{40\times 40}$ (left) and its inverse $\mathbf{Q}^{-1}$ (right).

GMRES(m)

Convergence bounds for iterative solvers?

✓ Hilbert space setting

- Symmetric case: bounded $\kappa_S \Rightarrow$ Bounded linear convergence rate ✓
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GMRES(m)

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- Application of the weighted (resp. Euclidean) restarted GMRES(m) to $\mathbf{Qx} = \mathbf{d}$.
- We arrive at the residuals:

$$\|\mathbf{r}_k\|_H := \|\mathbf{d} - \mathbf{Q}\mathbf{x}_k\|_H = \min_{\mathbf{x}\in\mathcal{K}^k(\mathbf{Q},\mathbf{r}_0)} \|\mathbf{d} - \mathbf{Q}\mathbf{x}\|_H,$$

$$\|\tilde{\mathbf{r}}_k\|_2 := \|\mathbf{d} - \mathbf{Q}\tilde{\mathbf{x}}_k\|_2 = \min_{\mathbf{x}\in\mathcal{K}^k(\mathbf{Q},\mathbf{r}_0)} \|\mathbf{d} - \mathbf{Q}\mathbf{x}\|_2$$

GMRES(m)

Lemma (Weighted GMRES(m): Linear bounds[1])

Let $\mathbf{Q} \in \mathbb{C}^N$, *with* $0 < \mathcal{V}_H(\mathbf{Q})$ *and set* $1 \leq m \leq N$. *Then, the* k*-th residual of weighted GMRES*(m) *for* $1 \leq k \leq N$ *satisfies:*

$$\frac{\|\mathbf{r}_k\|_H}{\|\mathbf{r}_0\|_H} \leq \left(1 - \mathcal{V}_H(\mathbf{Q})\mathcal{V}_H\left(\mathbf{Q}^{-1}\right)\right)^{\frac{k}{2}}$$

with

$$\mathcal{V}_H(\mathbf{Q}) := \min_{\mathbf{u}\in\mathbb{C}^N\setminus\{\mathbf{0}\}} \frac{|(\mathbf{Q}\mathbf{u}, \mathbf{u})_H|}{(\mathbf{u}, \mathbf{u})_H}$$

[1] P. Escapil-Inchauspé and C. Jerez-Hanckes, *Bi-Operator Preconditioning*, Computers & Mathematics with Applications, 2021.

GMRES(m) for OP-PG - Assumption

Assumption (Assumption 1)

For OP-PG with $X := H$ *being a Hilbert space with inner product* $(\cdot,\cdot)_H$*, we assume that* $\mathsf{P}_h\mathsf{A}_h$ *and its inverse satisfy*

$$\frac{\gamma_{\mathsf{C}}\gamma_{\mathsf{A}}}{\|\mathsf{m}\|\|\mathsf{n}\|} \leq \mathcal{V}_H(\mathsf{P}_h\mathsf{A}_h) \quad \textit{and} \quad \frac{\gamma_{\mathsf{M}}\gamma_{\mathsf{N}}}{\|\mathsf{c}\|\|\mathsf{a}\|} \leq \mathcal{V}_H((\mathsf{P}_h\mathsf{A}_h)^{-1})$$

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Theorem (GMRES(m): Linear convergence estimates for OP-PG[1])

Consider OP-PG with $X =: H$ Hilbert and $(\cdot,\cdot)_H$ such that Assumption 1 holds. Then, GMRES(m) for $1 \leq k, m \leq N$ leads to

$$\Theta_k^{(m)} \leq \left(1 - \frac{1}{\mathsf{K}_\star}\right)^{\frac{1}{2}} \quad \text{and} \quad \widetilde{\Theta}_k^{(m)} \leq \mathsf{K}_{\Lambda_h}\left(1 - \frac{1}{\mathsf{K}_\star}\right)^{\frac{1}{2}}$$

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- h-independent convergence for weighted GMRES(m)
- Offset factor K_{Λ_h} for Euclidean GMRES(m)

Assumption (Assumption 2)

For Bi-Parametric OP-PG with $X := H$ *being a Hilbert space with inner product* $(\cdot,\cdot)_H$*, we assume that* $\mathsf{P}_{h,\mu}\mathsf{A}_{h,\nu}$ *and its inverse satisfy*

$$\frac{\gamma_{\mathsf{C}_\mu}\gamma_{\mathsf{A}_\nu}}{\|\mathsf{m}\|\|\mathsf{n}\|} \leq \mathcal{V}_H(\mathsf{P}_{h,\mu}\mathsf{A}_{h,\nu}) \quad \textit{and} \quad \frac{\gamma_{\mathsf{M}}\gamma_{\mathsf{N}}}{\|\mathsf{c}_\mu\|\|\mathsf{a}_\nu\|} \leq \mathcal{V}_H((\mathsf{P}_{h,\mu}\mathsf{A}_{h,\nu})^{-1})$$

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GMRES(m) for Bi-Parametric OP-PG - Assumption

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For Bi-Parametric OP-PG with $X := H$ being a Hilbert space with inner product $(\cdot,\cdot)_H$, we assume that $\mathsf{P}_{h,\mu}\mathsf{A}_{h,\nu}$ and its inverse satisfy

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Theorem (GMRES(m): Linear convergence estimates for Bi-Parametric OP-PG[1])

Consider Bi-Parametric OP-PG with $X =: H$ Hilbert and $(\cdot,\cdot)_H$ such that Assumption 2 holds. Then, GMRES(m) for $1 \leq k, m \leq N$ leads to

$$\Theta_k^{(m)} \leq \left(1 - \frac{1}{\mathsf{K}_{\star,\mu,\nu}}\right)^{\frac{1}{2}} \quad \text{and} \quad \widetilde{\Theta}_k^{(m)} \leq \mathsf{K}_{\Lambda_h}\left(1 - \frac{1}{\mathsf{K}_{\star,\mu,\nu}}\right)^{\frac{1}{2}}$$

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Discussion

Theorem (GMRES(m): Linear convergence estimates for Bi-Parametric OP-PG[1])

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- Controlled convergence rates for GMRES(m) with respect to (μ, ν)-perturbations
- Bounded $\mu, \nu = \mathcal{O}(1)$ guarantee convergence for weighted GMRES(m) (and Euclidean GMRES(m) up to a K_{Λ_h}-term for $\mathsf{K}_{\Lambda_h} < 1$)

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Discussion

Theorem (GMRES(m): Linear convergence estimates for Bi-Parametric OP-PG[1])

Consider Bi-Parametric OP-PG with $X =: H$ *Hilbert and* $(\cdot,\cdot)_H$ *such that Assumption 2 holds. Then, GMRES(m) for* $1 \leq k, m \leq N$ *leads to*

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- Controlled convergence rates for GMRES(m) with respect to (μ,ν)-perturbations
- Bounded $\mu,\nu = \mathcal{O}(1)$ guarantee convergence for weighted GMRES(m) (and Euclidean GMRES(m) up to a K_{Λ_h}-term for $\mathsf{K}_{\Lambda_h} < 1$)
- Super-linear convergence results

[1] P. Escapil-Inchauspé and C. Jerez-Hanckes, *Bi-Operator Preconditioning*, Computers & Mathematics with Applications, 2021.

- *Carleman Class* $C^p(H)$ for $p > 0$:

$$|||\mathsf{K}|||_p = \|\sigma(\mathsf{K})\|_p := \left(\sum_{i=1}^{\infty} \sigma_i(\mathsf{K})^p\right)^{1/p} < \infty$$

- Q is a p-*class Fredholm operator of the second-kind* if

$$\mathsf{Q} - \mathsf{I} =: \mathsf{K} \in \mathcal{C}^p(H) \tag{2}$$

Define the commuting diagram

$$(\!(\mathsf{CA})\!)^p_{\mu,\nu} : \quad \begin{array}{ccc} H & \xrightarrow{\mathsf{A}_\nu} & Y' \\ {\scriptstyle \mathsf{I}^{-1}}\uparrow & & \downarrow{\scriptstyle \mathsf{N}^{-1}} \\ H & \xleftarrow[\mathsf{C}_\mu]{} & V \end{array} .$$

Theorem (GMRES: Super-linear convergence estimates for $((\mathsf{CA}))^{p}_{\mu,\nu}$ [1])

Consider $((\mathsf{CA}))^{p}_{\mu,\nu}$ for any $p \geq 0$ and define $\mathsf{K}_{\mu,\nu} := \mathsf{C}_{\mu}\mathsf{N}^{-1}\mathsf{A}_{\nu} - \mathsf{I} \in \mathcal{C}^{p}(H)$. Then, for weighted and Euclidean GMRES, respectively, it holds that

$$\begin{aligned} \Theta_k &\leq \frac{\|\mathsf{n}\|}{\gamma_{\mathsf{C}}\gamma_{\mathsf{A}}\gamma_{\mathsf{M}}} \frac{\overline{\sigma}_k(\mathsf{K}_{\mu,\nu})}{(1-\mu)(1-\nu)} \\ &\leq \frac{\|\mathsf{n}\|}{\gamma_{\mathsf{C}}\gamma_{\mathsf{A}}\gamma_{\mathsf{M}}} \frac{|||\mathsf{K}_{\mu,\nu}|||_p}{(1-\mu)(1-\nu)} k^{-\frac{1}{p}} \quad \text{if} \quad p > 0, \end{aligned} \tag{3}$$

and

$$\begin{aligned} \widetilde{\Theta}_k &\leq \mathsf{K}_{\Lambda_h} \frac{\|\mathsf{n}\|}{\gamma_{\mathsf{C}}\gamma_{\mathsf{A}}\gamma_{\mathsf{M}}} \frac{\overline{\sigma}_k(\mathsf{K}_{\mu,\nu})}{(1-\mu)(1-\nu)} \\ &\leq \mathsf{K}_{\Lambda_h} \frac{\|\mathsf{n}\|}{\gamma_{\mathsf{C}}\gamma_{\mathsf{A}}\gamma_{\mathsf{M}}} \frac{|||\mathsf{K}_{\mu,\nu}|||_p}{(1-\mu)(1-\nu)} k^{-\frac{1}{p}} \quad \text{if} \quad p > 0. \end{aligned} \tag{4}$$

- ✓ Weighted GMRES **converges super-linearly**
- ✓ Euclidean GMRES **can converge super-linearly** (e.g., for bounded K_{Λ_h})
- ✓ **Exhaustive** and **controlled** convergence results for GMRES

[1] P. Escapil-Inchauspé and C. Jerez-Hanckes, *Bi-Operator Preconditioning*, Computers & Mathematics with Applications, 2021.

Conclusions and Future Work

1. Explicit bounds on GMRES and CG for OP-PG linear systems
2. Super-linear convergence explained for Carleman class operators
3. Extension possible to more general operator cases following Blechta '21
4. Justification for rough offline preconditioners
5. *h*-optimal DeepONets-based OP

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Thank you

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