# Multi-Domain Solutions of PDEs Posed on Perforated Domains

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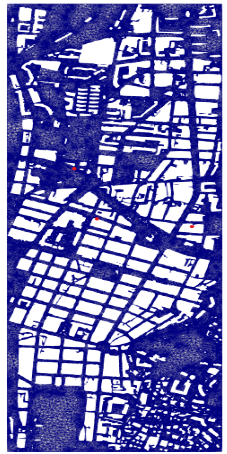
Preconditioning, Atlanta, 11 June 2024

# Motivation and model problem

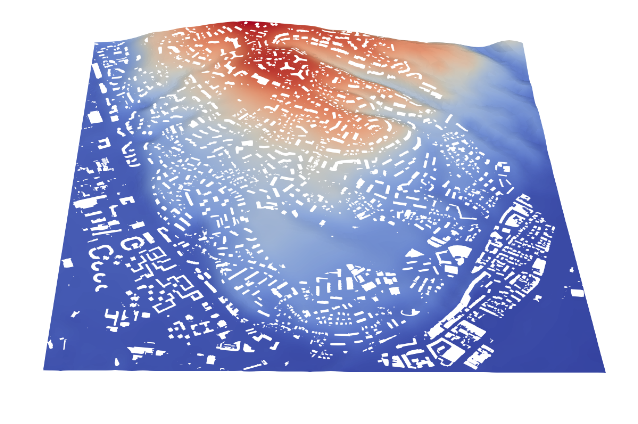
# Motivation: Urban Flood Modeling

- Efficiently solve problems on perforated domains.
- Expect corner singularities, triangle of varying magnitude, many degrees of freedom.



# Modelling a Realistic Problem

- Realistic topography ($z_b(x, y)$ of Nice, France);
- Rainfall data (source term): Can be taken from previous flood events (rain gauge data);
- Flood maps: previous areas of flood risk.



# Nonlinear Problem: Diffusive Wave model and discretisation

## A non linear problem

$$\begin{cases} \partial_t u + \operatorname{div} \mathcal{F}(x,u,\nabla u) &= f, \text{ in } \Omega, \\ \mathcal{F}(x,u,\nabla u)\cdot \mathbf{n} &= 0, \text{ on } \partial\Omega \cap \partial\Omega_S, \\ u &= g, \text{ on } \partial\Omega \setminus \partial\Omega_S. \end{cases}$$

$$\mathcal{F}(x,u,\nabla u) = c_f \frac{h(u, z_b(\mathbf{x}))^\alpha}{||\nabla u||^{1-\gamma}} \nabla u,$$

- $z_b(\mathbf{x})$: Bathymetry;
- $h(u, z_b(\mathbf{x})) = \max(u - z_b(\mathbf{x}), 0)$: Water depth;
- $\alpha > 1, 0 < \gamma \leq 1$.
- $c_f > 1$ : friction coefficient.

## Discretisation

Discretisation in space and time:

$$F(\mathbf{u}) := \frac{P}{\Delta t}(\mathbf{u} - \mathbf{u}^{old}) + K(\mathbf{u}) = 0, \tag{1}$$

where P is the (lumped) mass-matrix.

- Backward-Euler for time discretisation;
- $K(\mathbf{u})$ is discretisation of nonlinear term (FEM/FVM) and source term;
- Perform upwinding on $h(u, z_b(\mathbf{x}))^\alpha$ term (due to degeneracy);
- Adaptive time-stepping may be necessary for Newton's method on this system.

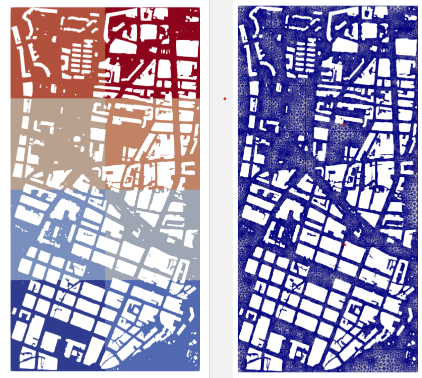
## The purpose of this work

- Design a fast solution method for the **nonlinear multiscale** problem $F(\mathbf{u}) = 0$.
- Simulate the whole time-dependent problem.

# Overlapping Schwarz: linear and nonlinear preconditioners

### Divide and conquer

- Partition of domain $\Omega$ into subdomains $\{\Omega_j\}_{j=1}^N$;
- Two levels of discretisation: 'Coarse' and 'Fine';
- Local subdomain solves can be done in parallel;
- Schwarz methods: use overlapping subdomains.



**Idea**: Solve model problem on each subdomain locally, with boundary conditions taken from adjacent subdomains.

# Linear preconditioning: Newton-Krylov-Schwarz (NKS)

**Newton's method to solve $F(\mathbf{u}) = 0$**

Given initial $\mathbf{u}^0$, for outer iteration $n = 0, \dots,$ to convergence,

- Solve for $\delta^n : \nabla F(\mathbf{u}^n)\delta^n = F(\mathbf{u}^n)$,
- Update $\mathbf{u}^{n+1} = \mathbf{u}^n - \delta^n$.

At each Newton's iteration we need to solve a linear system:

$$J_n \delta_n = F_n$$

with $J_n = \nabla F(\mathbf{u}^n)$, $F_n = F(\mathbf{u}^n)$.

**DD preconditioning**

Solve the preconditioned system

$$M^{-1} J_n \delta_n = M^{-1} F_n,$$

by a Krylov method, for some domain decomposition preconditioner $M^{-1}$.

- Does not change convergence/robustness of Newton's method.
- $M^{-1} \approx J_n^{-1} \rightarrow$ improves convergence of linear Krylov solver.

X.-C. Cai, W. D. Gropp, D. E. Keyes, and M. D. Tidriri, Newton-Krylov-Schwarz methods in CFD, 1994

# Nonlinear Preconditioning

**Goal**

Instead of $F(\mathbf{u}) = 0$, solve $N(F(\mathbf{u})) = 0$ via Newton.

- $N(v) = 0 \to v = 0$;
- $N(F(v))$ straightforward to compute.

**Use a fixed point iteration**

$$\mathbf{u}^{n+1} = P(\mathbf{u}^{n}), \tag{2}$$

solve $\mathcal{F}(\mathbf{u}) = P(\mathbf{u}) - \mathbf{u} = 0 \rightsquigarrow \mathcal{F}(\mathbf{u}) = 0$ is the preconditioned nonlinear system.

Nonlinear preconditioning can **improve convergence/robustness** of Newton's method and **localise** difficult nonlinearities.

X.-C. Cai and D. E. Keyes, Nonlinearly preconditioned inexact newton algorithms, SISC (2002)

# Nonlinear Restricted Additive Schwarz (NRAS) : $F(\mathbf{u}) = 0$

## Idea: use nonlinear Schwarz

- Decomposition into subdomains $\Omega_j$
- Start from an initial guess $\mathbf{u}^0$
- Perform local nonlinear subdomain solves

$$R_j F(R_j^T G_j(\mathbf{u}^n) + (I - R_j^T R_j)\mathbf{u}^n)) = 0.$$

  where $R_j$ are restriction operators and $D_j$ partition of unity matrices.
- "Glue" together local solutions $G_j(\mathbf{u}^n)$

$$\mathbf{u}^{n+1} = \sum_j R_j^T D_j G_j(\mathbf{u}^n)$$

## Advantages

- Local subproblems are solved via Newton with negligible cost;
- Local solves can be done in parallel.
- Natural nonlinear solver based on the decomposition into subdomains.

**Downsides**: this is the non-linear equivalent of the iterative version of RAS, hence in general with a slow convergence.

# Restricted Additive Schwarz Exact Newton (RASPEN)

## From NRAS to RASPEN

- Start with the nonlinear fixed point iteration

$$\mathbf{u}^{n+1} = \sum_j R_j^T D_j G_j(\mathbf{u}^n)$$

which solves the nonlinear system $\mathcal{F}(\mathbf{u}) := \sum_j R_j^T D_j G_j(\mathbf{u}) - \mathbf{u} = 0$.

- Accelerate via Newton (the equivalent of GMRES in the nonlinear world) $\rightsquigarrow$ the RASPEN method.

## Main features

- **Acceleration** of convergent fixed point iteration;
- Computation of **exact Jacobian** $\nabla\mathcal{F}$, or specifically the matrix-vector product $\nabla\mathcal{F}v$ for some v.

$$\nabla\mathcal{F}(\mathbf{u}^n) = \nabla(\mathbf{u}^n - \sum_j R_j^T D_j G_j(\mathbf{u}^n))$$

$$= \sum_j R_j^T D_j [R_j \nabla F(\mathbf{u}^n) R_j^T]^{-1} R_j \nabla F(\mathbf{u}^n)$$

V. Dolean, M. J. Gander, W. Kheriji, F. Kwok, and R. Masson, Nonlinear preconditioning: How to use a nonlinear schwarz method to precondition newton's method, SISC (2016).

# One-level RASPEN

The algorithm, for each time step, is given by:

**One-level algorithm**

Given initial $\mathbf{u}^0$, for outer iteration $n = 0, \ldots,$ do until convergence

- Solve local subproblems $R_j F(R_j^T G_j(\mathbf{u}^n) + (I - R_j^T R_j)\mathbf{u}^n)) = 0$ for $G_j(\mathbf{u}^n)$ (Newton);
- Glue local solutions $\widehat{\mathbf{u}}^n = \sum_j R_j^T D_j G_j(\mathbf{u}^n)$;
- Set $\mathcal{F}(\mathbf{u}) = \mathbf{u} - \widehat{\mathbf{u}}^n$;
- Solve $\nabla \mathcal{F}(\mathbf{u}^n) \boldsymbol{\Delta}^{\mathbf{n}} = \mathcal{F}(\mathbf{u}^n)$.
- Update $\mathbf{u}^{n+1} = \mathbf{u}^n - \boldsymbol{\Delta}^{\mathbf{n}}$.

V. Dolean, M. J. Gander, W. Kheriji, F. Kwok, and R. Masson, Nonlinear preconditioning: How to use a nonlinear schwarz method to precondition newton's method, SISC (2016).

# Need for coarse correction

**Advantages of coarse spaces**

- Allow for global communication between all subdomains.
- Are necessary for scalability for large number of subdomains for preconditioning of linear/nonlinear problems.

**Aim**

- Robustness with respect to perforation size/location (even along subdomain interfaces);
- Robustness with respect to the number of subdomains N.

Two options:

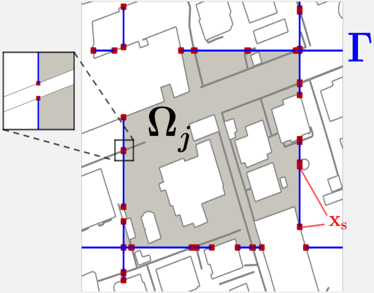
- Coarse space based on the FAS like in Dolean et al. (2016)
- Here, we choose the coarse space specially tailored to perforated domains.

V. Dolean, M. J. Gander, W. Kheriji, F. Kwok, and R. Masson, Nonlinear preconditioning: How to use a nonlinear Schwarz method to precondition newton's method, SISC (2016).

# The construction of the coarse space

# Coarse grid nodes for coarse space basis functions

- Coarse grid nodes arise at the intersection of nonoverlapping skeleton with a perforation boundary;
- $(\phi_s)_{s\in\{1,\dots,N_x\}}$ : Locally harmonic basis functions for each coarse grid node.
- $\#$ of coarse grid nodes is automatically generated.
- Continuously, the coarse space is given by $V_H = \text{span}\{\phi_s\}$.
- Think of as 'enriching' Multi-scale FEM (MsFEM) coarse space.

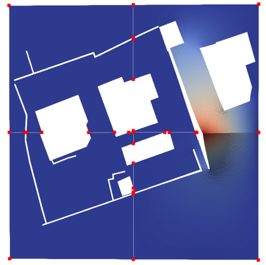



M. Boutilier, K. Brenner, and V. Dolean, Robust methods for multiscale coarse approximations of diffusion models in perforated domains, APNUM, 2024.

# Basis functions: Harmonic local solutions

For all nonoverlapping $\left(\Omega'_j\right)_{j\in\{1,\dots,N\}}$ and $s = 1, \dots, N_{\mathbf{x}}$, to obtain $\phi_{s,j} = \phi_s|_{\Omega_j}$, solve

$$\begin{cases} \Delta\phi_{s,j} = 0 & \text{in} \quad \Omega'_j, \\ \dfrac{\partial\phi_{s,j}}{\partial n} = 0 & \text{on} \quad \partial\Omega'_j \cap \partial\Omega_S, \\ \phi_{s,j} = g_s & \text{on} \quad \partial\Omega'_j \setminus \partial\Omega_S. \end{cases}$$



$g_s : \Gamma \to [0,1]$ as: for $i = 1, \dots N_{\mathbf{x}}$,

$$g_s(\mathbf{x}_i) = \begin{cases} 1, & s = i, \\ 0, & s \neq i, \end{cases}$$

- $g_s$ is linearly extended on the remainder of $\Gamma$.
- Can also include higher-order polynomials on coarse edgres.

M. Boutilier, K. Brenner, and V. Dolean, Robust methods for multiscale coarse approximations of diffusion models in perforated domains, APNUM, 2024.

# Two-level RASPEN with "multiscale" coarse space

We add the coarse correction multiplicatively and discretely.

**Two-level algorithm**

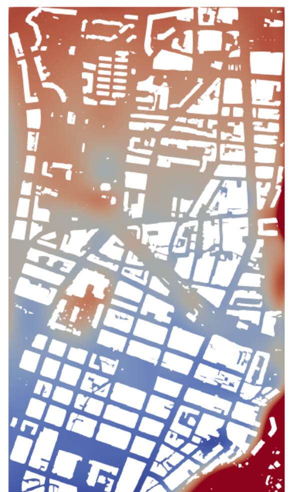
Given initial $\mathbf{u}^0$, for outer iteration $n = 0, \dots,$ do until convergence

- Solve local subproblems $R_j F(R_j^T G_j(\mathbf{u}^n) + (I - R_j^T R_j)\mathbf{u}^n)) = 0$ for $G_j(\mathbf{u}^n)$ (Newton);
- Glue local solutions $\widehat{\mathbf{u}}^n = \sum_j R_j^T D_j G_j(\mathbf{u}^n)$;
- Set $\mathcal{F}(\mathbf{u}) = \mathbf{u} - \widehat{\mathbf{u}}^n$;
- Solve coarse problem $R_0 F(\widehat{\mathbf{u}}^n - R_0^T c_0^n) = 0$ for $c_0^n$;
- Set $\mathcal{F}(\mathbf{u}^n) = \mathbf{u}^n - \widehat{\mathbf{u}}^n + R_0^T c_0^n$;
- Solve $\nabla \mathcal{F}(\mathbf{u}^n) \boldsymbol{\Delta}^n = \mathcal{F}(\mathbf{u}^n)$.
- Update $\mathbf{u}^{n+1} = \mathbf{u}^n - \boldsymbol{\Delta}^n$.

# Numerical Results

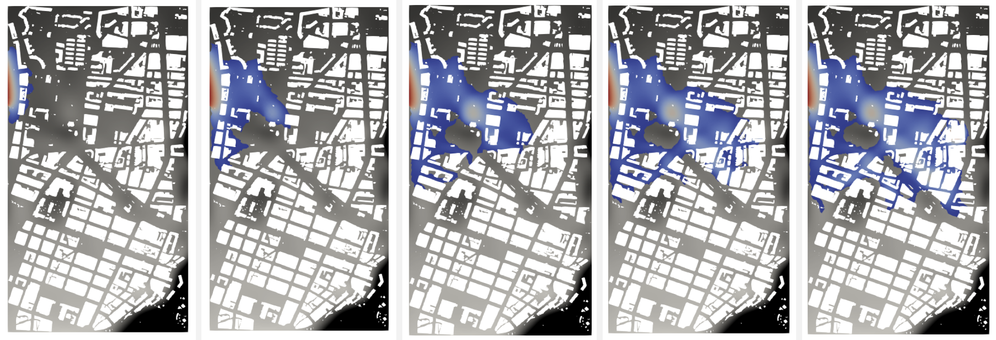
# Setup example model problem

- Excessive water flow coming from Paillon river in Nice, France;
- Dirichlet boundary conditions with initial condition $u_0 > z_b$ at leftmost boundary (river).
- $\alpha = \frac{3}{2}, \gamma = 1$, 0 source term, $c_f = 30$.

# Solution over time

- $z_b$: black and white (darker = higher elevation)
- h (water depth): colour



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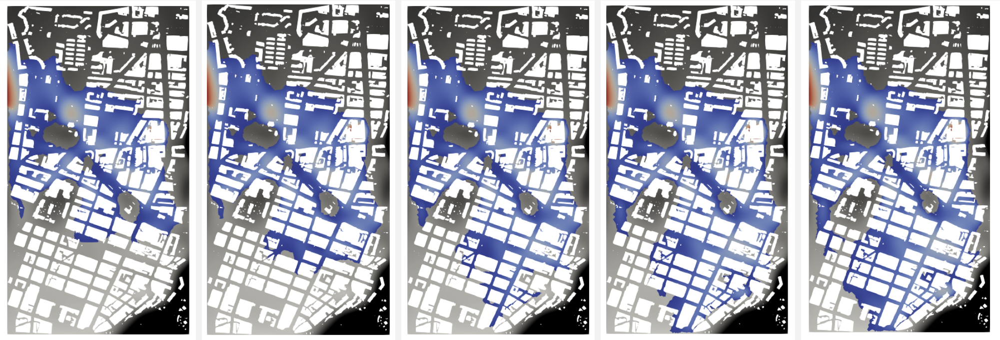


# Linearised Newton System

We can also perform typical linear DD methods on the linearised system at each Newton iteration (SNK: Schwarz-Newton-Krylov)

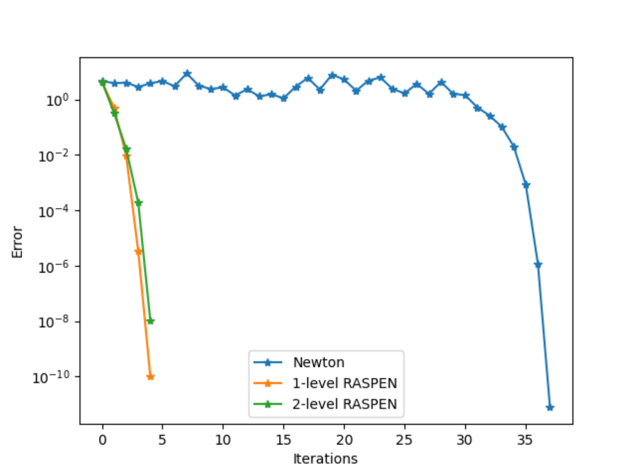
$$M_{RAS,2}^{-1} J_n \mathbf{u} = M_{RAS2}^{-1} \mathbf{F_n}.$$

where

$$M_{RAS,2}^{-1} = \sum_{j=1}^{N} R_j^T D_j (R_j J_n R_j^T)^{-1} R_j + R_0^T (R_0 J_n R_0^T)^{-1} R_0.$$

**Question**: Will our coarse space (designed for Poisson equation) work well for the linearized Newton system? $\rightsquigarrow$ Perform scalability tests.

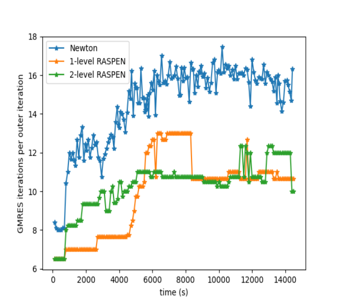
# Numerical Results: First time step



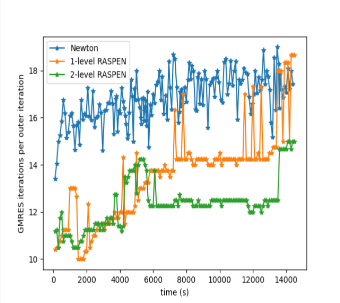

- 1 and 2 level RASPEN immediately enter region of quadratic convergence, quite insensitive to initial guess.

# Numerical results: GMRES iterations

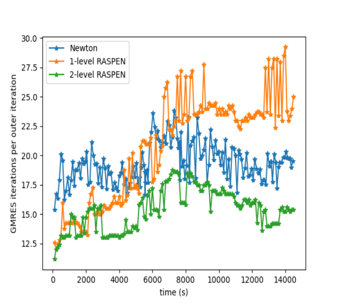
Average GMRES iterations per outer iteration for each time step.




$N = 2 \times 2$




$N = 4 \times 4$




$N = 8 \times 8$

# Numerical results: Outer iteration, GMRES iterations, and convergence curves

| $N = 2 \times 2$ | T. steps | Outer Its | GMRES Its |
| --- | --- | --- | --- |
| Newton | 160 | 2216 | 19497 |
| 1-level RASPEN | 146 | 440 | 2630 |
| 2-level RASPEN | 146 | 532 | 3187(+1300 coarse) |

| $N = 4 \times 4$ | T. steps | Outer Its | GMRES Its |
| --- | --- | --- | --- |
| Newton | 160 | 2159 | 23989 |
| 1-level RASPEN | 146 | 554 | 5026 |
| 2-level RASPEN | 146 | 576 | 4507(+1719 coarse) |

| $N = 8 \times 8$ | T. steps | Outer Its | GMRES Its |
| --- | --- | --- | --- |
| Newton | 160 | 2266 | 30811 |
| 1-level RASPEN | 146 | 602 | 9668 |
| 2-level RASPEN | 146 | 609 | 5627(+2255 coarse) |

- 1-level RASPEN: GMRES iterations do not scale with N.

# Closing Remarks

- RASPEN is a very good alternative for our model problem, improving the convergence of Newton's method;
- Local time step reduction can be employed for problem subdomains to avoid a global time step reduction;
- For a larger number of subdomains, a coarse correction is necessary for scalability $\rightarrow$ our coarse space designed for the Poisson equation works well in this nonlinear case;
- As an alternative, the two-level RAS preconditioner with our coarse space provides a scalable number of Krylov iterations.

M. Boutilier, K. Brenner, and V. Dolean, Two-level Nonlinear Preconditioning Methods for Flood Models Posed on Perforated Domains, arXiv:2406.06189, 2024.

# Funding Acknowledgement

This work has been supported by ANR Project Top-up (ANR-20-CE46-0005).

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Thanks for your attention!