Robust domain decomposition methods for high-contrast multiscale problems on irregular domains

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International Conference on Preconditioning Techniques for Scientific and Industrial Applications
Georgia Institute of Technology
June 10, 2024

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Outline

- Introduction
- Virtual elements
- Domain Decomposition Methods
- Generalized eigenvalue problems
- Numerical experiments

Introduction

- Interest in problems posed in $H(curl)$ with irregular subdomains

  J.C., *Domain Decomposition Methods for Problems in $H(curl)$*, PhD Thesis, NYU 2015. Advisor: Prof. Olof Widlund.

- Met Juan Galvis at the (virtual) Mathematical Congress of the Americas 2021 (Session: Applied Math and Computational Methods and Analysis across the Americas).

Introduction

- For simplicity, consider the problem

$$-\text{div}(\kappa \nabla u) = f, \quad \boldsymbol{x} \in D \subset \mathbb{R}^2,$$

  with homogeneous Dirichlet boundary conditions
- The coefficient $\kappa = \kappa(x)$ represents the permeability of the porous media D
- Problem: Find $u \in H_0^1(D)$ such that

$$a(u, v) := \int_\Omega \kappa \nabla u \cdot \nabla v \; d\boldsymbol{x} = (f, v)_{0,D} \quad \forall \; v \in H_0^1(D)$$

Two ideas:

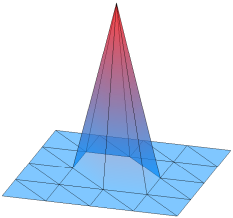
- Given the contrast $\eta = \max \kappa(x) / \min \kappa(x)$, obtain bounds that are independent of κ (work by Juan Galvis)
- Handle irregular decompositions/interfaces (work by J.C.)

Discretization

- Consider a polygonal mesh and virtual elements.
- Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h) \quad \forall\ v_h \in V_h.$$

- In order to solve the associated linear system we build a preconditioner and use PCG.
- For simplicity we consider a two-level overlapping additive Schwarz



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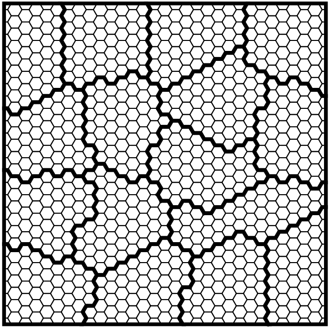

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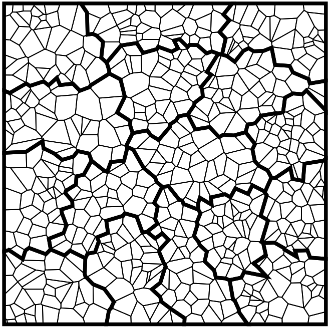


Virtual elements

B. Ahmad, A. Alsaedi, F. Brezzi, L. D. Marini, and A. Russo, *Equivalent projectors for virtual element methods*, Comput. Math. Appl., 66 (2013), pp. 376–391.

L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. D. Marini, and A. Russo, *Basic principles of virtual element methods*, Math. Models Methods Appl. Sci., 23 (2013), pp. 199–214.

L. Beirão da Veiga, F. Brezzi, L. D. Marini, and A. Russo, *The hitchhiker's guide to the virtual element method*, Math. Models Methods Appl. Sci., 24 (2014), pp. 1541–1573.

Virtual elements

- For any subdomain Ω_i, let

$$\mathcal{B}_1(\partial E) := \{v \in C^0(\partial E) : v|_e \in \mathcal{P}_1(e) \ \forall \ e \subset \partial E\},$$

  where e represents any edge on the boundary of Ω_i
- Local virtual space:

$$V_1^E := \{v \in H^1(E) : v|_{\partial E} \in \mathcal{B}_1(\partial E), \ \Delta v = 0\}.$$

- V_1^E is piecewise-linear on the boundary and harmonic in the interior, and its dof are the values at the vertices of the polygon

- Bilinear form:

$$a_h(u,v) = \sum_{E \in \mathcal{T}^h} \int_E \nabla \Pi^{\nabla}_{E,1} u \cdot \nabla \Pi^{\nabla}_{E,1} v +$$

$$\sum_{r=1}^{N^E_{\text{dof}}} \text{dof}_r((I - \Pi^{\nabla}_{E,1})u)\text{dof}_r((I - \Pi^{\nabla}_{E,1})v)$$

- $\Pi^{\nabla}_{E,1} : V_1(E) \to \mathcal{P}_1(E)$ is the L^2 projection of the gradient defined as the linear polynomial $w = \Pi^{\nabla}_{E,1} v$ such that

$$\int_E \nabla w \cdot \nabla p = \int_E \nabla v \cdot \nabla p \quad \forall\ p \in \mathcal{P}_1(E)$$

and

$$\frac{1}{N^E_V} \sum_{i=1}^{N^E_V} v(\boldsymbol{x}_i) = \frac{1}{N^E_V} \sum_{i=1}^{N^E_V} w(\boldsymbol{x}_i).$$

Discretization

- Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h) \quad \forall\, v_h \in V_h.$$

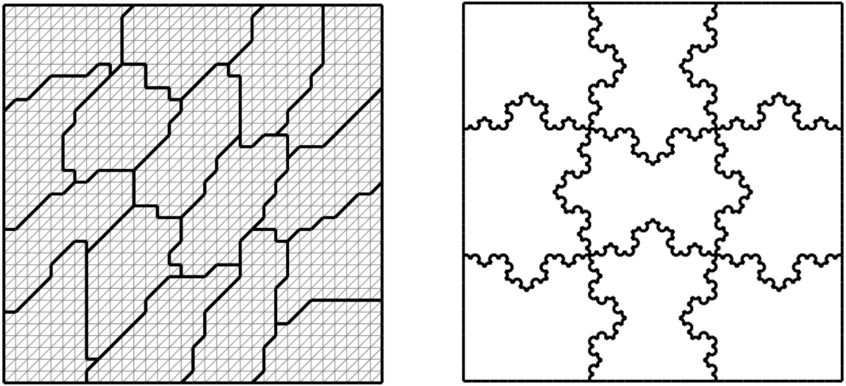
- If $V_h = \langle \varphi_i \rangle_i$, the problem is rewritten as

$$Ax = b$$

with $(A)_{ij} = a_h(\phi_j, \phi_i)$, $(b)_j = (f, \Pi\phi_j)$ and x is the vector of coordinates of the solution u

Domain Decomposition

- Efficient parallel computing algorithms
- How to define coarse functions for irregular subdomains?



Some previous work

- For scalar elliptic problems in H^1 with irregular subdomains:

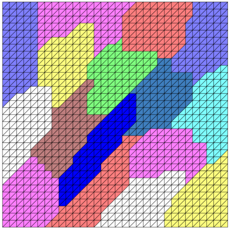
C. R. Dohrmann, A. Klawonn, and O. B. Widlund, *Domain decomposition for less regular subdomains: Overlapping Schwarz in two dimensions*, SIAM J. Numer. Anal. 2008.

O. B. Widlund, *Accommodating irregular subdomains in domain decomposition theory*, 2009.

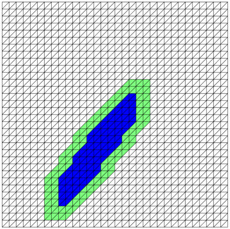
C. R. Dohrmann and O. B. Widlund, *An alternative coarse space for irregular subdomains and an overlapping Schwarz algorithm for scalar elliptic problems in the plane*, SIAM J. Numer. Anal., 2012.

Domain Decomposition - Overlapping version

- We divide the domain Ω into subdomains $\{D_i\}_{i=1}^N$
- Then construct overlapping subdomains $\{D_i'\}_{i=1}^N$



Subdomains D_i



Overlapping subdomains D_i'

METIS Subdomains for a triangular mesh

DD - Local solvers in D_i'

- Homogeneous Dirichlet problems in Ω_i', $i = 1, 2, \ldots, N$.
- Local spaces:

$$V_i := H_0^1(\Omega_i') \cap V_h$$

- Zero extension operator $R_i^T : V_i \to V_h$
- Exact solvers: $\tilde{A}_i = R_i A R_i^T$
- $\tilde{A}_i$: block of A that corresponds to the interior nodes of Ω_i'

Two-level Preconditioner

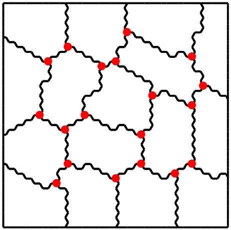
- Suppose we have a coarse space V_0 (with just a few dof per subdomain) and an operator $R_0^T : V_0 \to V_h$
- Exact solvers: $\tilde{A}_0 = R_0 A R_0^T$
- Preconditioner:

$$A_{ad}^{-1} = R_0^T \tilde{A}_0^{-1} R_0 + \sum_{i=1}^{N} R_i^T \tilde{A}_i^{-1} R_i$$

- Question: How to define a proper coarse space V_0 and $R_0^T : V_0 \to V_h$ for irregular subdomains?

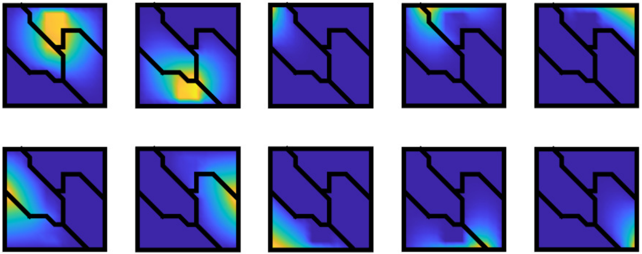
Partition of unity for decomposition

- One coarse function per subdomain vertex
- Common approach: define values on interface and then use discrete harmonic extensions.

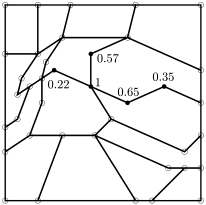


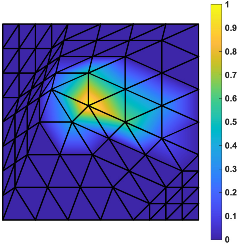
$\{\Omega_i\}_{i=1}^N$

Partition of unity



Partition of unity - values on interface





Partition of unity - extensions

- Usually, boundary values are extended as harmonic extensions to the interior of the subdomains.

J.C., *On the approximation of a virtual coarse space for Domain Decomposition Methods in two dimensions.*, Mathematical Models and Methods in Applied Sciences, 2018.

- Partition of unity is constructed by computing projections to polynomial spaces of degree $k \geq 2$ in the interior of each subdomain
- No discrete harmonic extensions required

High-contrast coefficient

- For high-contrast elliptic problems in H^1 (regular subdomains):

Y. Efendiev, J. Galvis, and T. Hou. *Generalized multiscale finite element methods*, Journal of Computational Physics, 251:116–135, 2013.

J. Galvis and Y. Efendiev. *Domain decomposition preconditioners for multiscale flows in high contrast media.*, SIAM J. Multiscale Modeling and Simulation, 8:1461–1483, 2010.

J. Galvis, E.T. Chung, Y. Efendiev, and W. T. Leung. *On overlapping domain decomposition methods for high-contrast multiscale problems*, in International Conference on Domain Decomposition Methods, pages 45–57. Springer, 2017.

High-contrast coefficient

- We assume that

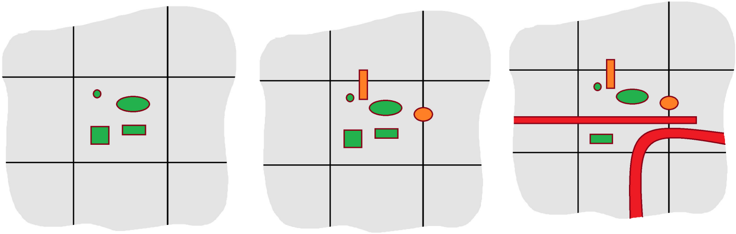
$$0 < \kappa_{\text{min}} \leq \kappa(x) \leq \kappa_{\text{max}} \; \forall x \in \overline{D}$$

- Define the contrast of κ restricted to Ω by

$$\eta_\Omega := \frac{\max_{x \in \overline{\Omega}} \kappa(x)}{\min_{x \in \overline{\Omega}} \kappa(x)}.$$

- It is known that the performance of iterative methods for the solution depend on η_D and on the local variations of κ across D

High-contrast multiscale coefficient



Example of a multiscale subdomain with interior high-contrast inclusions in green (left), boundary inclusions in orange (center), and long channels in red (right). We have $\kappa = 1$ in the gray background, and $\kappa \asymp \eta \gg 1$ inside the channels and inclusions.

High-contrast multiscale coefficient

- The label “multiscale coefficient” refers to the fact that the coefficient varies at multiple scales
- There is a large quantity of high-contrast subdomains scattered everywhere around the whole domain.

Theoretical analysis

- For the bound of the preconditioned system, we need a decomposition for a global field $v \in V = V^h(D)$ as

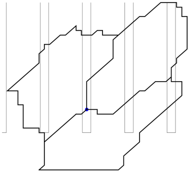
$$v = R_0^T v_0 + \sum_{j=1}^{N_S} R_j^T v_j \quad (v_i \in V_i). \tag{1}$$

- The decomposition (1) is stable in the sense that there exists $C_0 > 0$ such that

$$a_h(R_0^T v_0, R_0^T v_0) + \sum_{j=1}^{N_S} a_h(R_i^T v_i, R_i^T v_i) \leq C_0^2 a_h(v, v).$$

Theoretical analysis

- A global function $v \in V_h$ is restricted to ω_i, by identifying a local field $I_0^{\omega_i} v$ that will contribute to the coarse space



A subdomain vertex and the associated region ω_i

Theoretical analysis

- A global function $v \in V_h$ is restricted to ω_i, by identifying a local field $I_0^{\omega_i} v$ that will contribute to the coarse space
- A global coarse field can be assembled as

$$v_0 = I_0 v = \sum_{i=1}^{N_S} I^h(\chi_i(I_0^{\omega_i} v)), \tag{2}$$

  where I^h is the fine-scale nodal value interpolation.
- In classical two-level domain decomposition methods, $I_0^{\omega_i} v$ is the average of v in ω_i
- Note that in each coarse block $K \in \mathcal{T}^H$ we have

$$v - v_0 = \sum_{x_i \in K} I^h(\chi_i(v - I_0^{\omega_i} v)), \tag{3}$$

  where the sum goes over the subdomain vertices of K.

Theoretical analysis

- For each overlapping subdomain D'_j, the corresponding local part of the stable decomposition is defined by

$$v_j = I^h(\xi_j(v - v_0)),$$

with $v_0 = I_0 v$.

- $\{\xi_j\}_{j=1}^N$ is a partition of unity for the overlapping subdomains
- To bound the energy of v_j, we have that

$$\int_K \kappa |\nabla I^h(\xi_j(v - v_0))|^2 \preceq \int_{\omega_i} \kappa(\xi_j \chi_i)^2 |\nabla(v - I_0^{\omega_i} v)|^2 + \int_{\omega_i} \kappa |\nabla(\xi_j \chi_i)|^2 |v - I_0^{\omega_i} v|^2,$$

Theoretical analysis - previous works

- Poincaré inequality for the case of bounded coefficient combined with a small overlap trick; for high-contrast multiscale coefficients, the resulting bound depends on the contrast η in general
- L^∞ estimates:

$$\int_{\omega_i} \kappa |\nabla(\xi_j \chi_i)|^2 |v - I_0^{\omega_i} v|^2 \preceq ||\kappa|\nabla(\xi_j\chi_i)|^2||_\infty \int_{\omega_i} |v - I_0^{\omega_i} v|^2.$$

  Partitions of unity can be constructed such that the term $||\kappa|\nabla(\xi_j\chi_i)|^2||_\infty$ is bounded independently of the contrast
- **Local generalized eigenvalue problems**

Theoretical analysis

- We can write

$$\int_{\omega_i} \kappa |\nabla(\xi_j \chi_i)|^2 |v - I_0^{\omega_i} v|^2 \preceq \frac{1}{\delta^2 H^2} \int_{\omega_i} \kappa |(v - I_0^{\omega_i} v)|^2$$
$$\preceq C \int_{\omega_i} \kappa |\nabla v|^2,$$

where we need to justify the last inequality with constant independence of the contrast

- Consider the Rayleigh quotient

$$\mathcal{Q}(v) := \frac{\int_{\omega_i} \kappa |\nabla v|^2}{\int_{\omega_i} \kappa |v|^2}$$

with $v \in V^h(\omega_i)$.

Eigenvalue problem

- The associated eigenproblem is given by

$$-\text{div}(\kappa(x)\nabla\psi_\ell^{\omega_i}) = \lambda_\ell \kappa(x)\psi_\ell^{\omega_i} \text{ in } \omega_i,$$

  with homogeneous Neumann boundary conditions for floating subdomains and a mixed homogeneous Neumann-Dirichlet condition for subdomains that touch the boundary

- Write the spectrum as

$$\lambda_1 \le \lambda_2 \le \ldots \le \lambda_L < \lambda_{L+1} \le \ldots$$

  where $\lambda_1, ..., \lambda_L$ are small, asymptotically vanishing eigenvalues, and λ_L can be bounded below independently of the contrast.

Eigenvalue problem

Define the matrix A^{ω_i} corresponding to homogeneous Neumann problem by

$$v^T A^{\omega_i} w = \int_{\omega_i} \kappa \nabla v \cdot \nabla w \quad \text{for all } v, w \in V^h(\omega_i),$$

and the *modified mass matrix* of same dimension M^{ω_i} by

$$v^T M^{\omega_i} w = \int_{\omega_i} \kappa v w \quad \text{for all } v, w \in \widetilde{V}^h(\Omega), \tag{4}$$

where $\widetilde{V}^h = V^h(\omega_i)$ if $\overline{\Omega} \cap \partial D = \emptyset$ and $\widetilde{V}^h = \{v \in V^h(\omega_i) : v = 0 \text{ on } \partial\omega_i \cap \partial D\}$ otherwise. We have then the generalized eigenvalue problem

$$A^{\omega_i} \psi = \lambda M^{\omega_i} \psi. \tag{5}$$

Eigenvalue problem

- Given an integer L and $v \in V^h(\omega_i)$, we define

$$I_L^{\omega_i} v = \sum_{\ell=1}^{L} \left(\int_{\omega_i} \kappa v \psi_\ell^{\omega_i} \right) \psi_\ell^{\omega_i}. \tag{6}$$

- It is easy to prove that

$$\int_{\omega_i} \kappa (v - I_L^{\omega_i} v)^2 \leq \frac{1}{\lambda_{L+1}^{\omega_i}} a(v - I_L^{\omega_i} v, v - I_L^{\omega_i} v) \leq \frac{1}{\lambda_{L+1}^{\omega_i}} a(v, v). \tag{7}$$

- When $L = 1$, $\kappa = 1$ (or κ is smooth and bounded) and $\partial\omega_i$ is smooth (Lipschitz), we obtain the classical Poincaré inequality

Eigenvalue problem

- Define the set of coarse basis functions

$$\Phi_{i,\ell} = I^h(\chi_i \psi_\ell^{\omega_i}) \quad \text{for } 1 \leq i \leq N_c \text{ and } 1 \leq \ell \leq L_i,$$

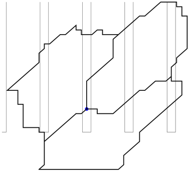
  where I^h is the fine-scale nodal value interpolation and L_i is an integer number for each $i = 1, \ldots, N_c$.
- Denote by V_0 the *local spectral multiscale* space

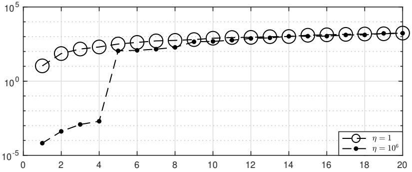
$$V_0 = \text{span}\{\Phi_{i,\ell} : 1 \leq i \leq N_c \text{ and } 1 \leq \ell \leq L_i\}.$$

- Define also the coarse interpolation $I_0 : V^h(D) \to V_0$ by

$$I_0 v = \sum_{i=1}^{N_c} \sum_{\ell=1}^{L_i} \left(\int_{\omega_i} \kappa v \psi_\ell^{\omega_i} \right) I^h(\chi_i \psi_\ell^{\omega_i}) = \sum_{i=1}^{N_c} I^h \left((I_{L_i}^{\omega_i} v) \chi_i \right),$$

Eigenvalues for high-contrast subdomains





(left) A subdomain vertex with three METIS subdomains. The coefficient κ is $\kappa = \eta$ inside the small rectangular channels, and $\kappa = 1$ in the background. (right) Eigenvalue distribution for $\eta = 1$ (circles) and $\eta = 10^6$ (black dots). The effect of having a high contrast coefficient implies the addition of four eigenfunctions, associated to the four eigenvalues smaller than 1.

Condition number

Theorem [J.C and J. Galvis (2023)]

$$cond(M_2^{-1}A) \preceq C_0^2 \preceq \max\left\{1 + \frac{1}{\delta^2 \lambda_{L+1}}, 1 + \frac{1}{H^2 \lambda_{L+1}}\right\},$$

where $\lambda_{L+1} = \min_{1 \leq i \leq N_c} \lambda_{L_i+1}^{\omega_i}$.

Condition number

Corollary [J.C and J. Galvis (2023)]

If we select L_i appropriately, it holds that

$$cond(M_2^{-1}A) \preceq C\left(1+\frac{H^2}{\delta^2}\right),$$

where C is independent of the contrast and the mesh size.

Numerical experiments

- We solve the resulting linear systems using a PCG method, to a relative residual tolerance of 10^{-6}.

Numerical examples - triangular mesh

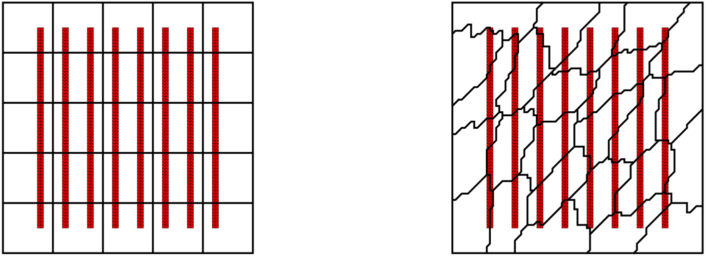


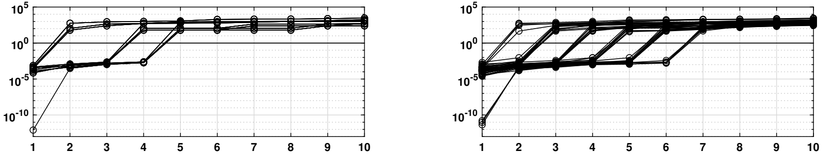
Figure: $\kappa = \eta \in \{1, 10^2, 10^4, 10^6\}$ for red elements, and $\eta = 1$ in the background for (left) square and (right) METIS subdomains

Numerical examples - triangular mesh

Subdomains	η	Non adaptive, harmonic			Adaptive, $k = 2$/harmonic		
		Cond	Iter	dimV0	Cond	Iter	dimV0
Squares	1e0	17.3	22	36	23.8/23.8	21/21	4
	1e2	44.5	36	36	26.8/21.0	28/26	56
	1e4	4827	87	36	5.0/6.3	18/18	96
	1e6	1.7e6	148	36	5.3/5.3	19/18	96
METIS	1e0	17.8	29	52	25.4/25.3	34/34	16
	1e2	31.3	45	52	28.4/12.7	43/27	96
	1e4	3741	167	52	6.9/6.0	23/22	190
	1e6	3.0e5	342	52	7.2/6.0	25/24	190

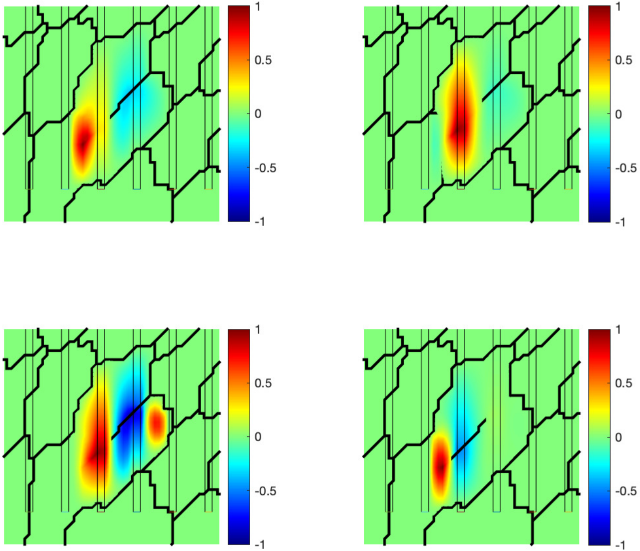
Number of iterations (Iter) and estimated condition number (Cond) for a triangular mesh with 12800 elements, 25 subdomains, $H/h \approx 16$, $H/\delta \approx 4$.

Numerical examples - triangular mesh

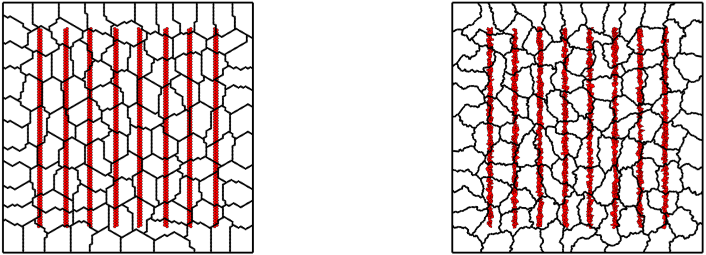


Smaller eigenvalues (circles) for (left) square and (right) METIS subdomains with a triangular mesh with 12800 elements and $\eta = 10^6$.

Numerical examples - triangular mesh

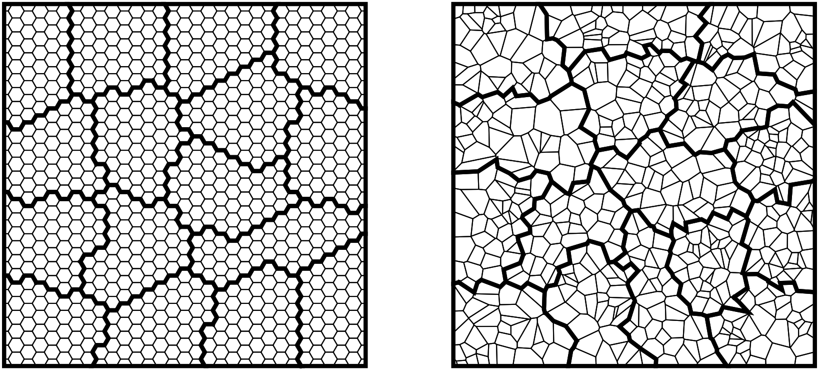


Numerical examples - hexagonal and Voroni meshes



$\kappa = \eta \in \{1, 10^2, 10^4, 10^6\}$ for red elements, and $\eta = 1$ in the background for (left) hexagonal and (right) Voronoi-type meshes with METIS subdomains.

Numerical examples - hexagonal and Voronoi meshes



Fine mesh and 16 subdomains (thick black lines) with METIS subdomains.

η	Non adaptive, harmonic			Adaptive, harmonic			Adaptive, $k = 2$		
	Cond	Iter	dimV0	Cond	Iter	dimV0	Cond	Iter	dimV0
$1e0$	29.6	35	202	52.1	48	96	47.7	48	96
$1e2$	34.3	45	202	20.8	37	191	52.5	61	191
$1e4$	1027	129	202	11.2	30	335	12.8	34	335
$1e6$	1.0e5	236	202	11.8	34	335	13.6	37	335

Number of iterations (Iter) until convergence of the PCG and condition number (Cond), for different values of the contrast η, with κ as shown in Figure 44, for an hexagonal mesh with 9699 elements, 100 subdomains, $H/h \approx 20$, $\delta \approx 2h$.

Numerical examples - Voroni mesh

η	Non adaptive, harmonic			Adaptive, harmonic			Adaptive, $k=2$		
	Cond	Iter	dimV0	Cond	Iter	dimV0	Cond	Iter	dimV0
1e0	43.6	42	202	62.7	54	97	68.3	55	97
1e2	45.5	48	202	21.1	37	199	63.5	53	199
1e4	1448	130	202	13.3	34	382	17.1	39	382
1e6	1.2e5	246	202	13.6	37	382	15.7	42	382

Number of iterations (Iter) until convergence of the PCG and condition number (Cond), for different values of the contrast η for a Voronoi mesh with 12325 elements, 100 subdomains, $H/h \approx 34$, $h \approx 0.136$, $\delta \approx 2h$.

Final remarks and future work

- Applications to different PDEs
- Particular interest for problems posed in $H(curl)$ and $H(div)$

J.C., J. Galvis, *Robust domain decomposition methods for high-contrast multiscale problems on irregular domains with virtual element discretizations*, Journal of Computational Physics, 2024.

Thank you!