Università degli Studi di Padova

Block triangular preconditioners for double saddle-point problems arising in mixed hybrid coupled poromechanics

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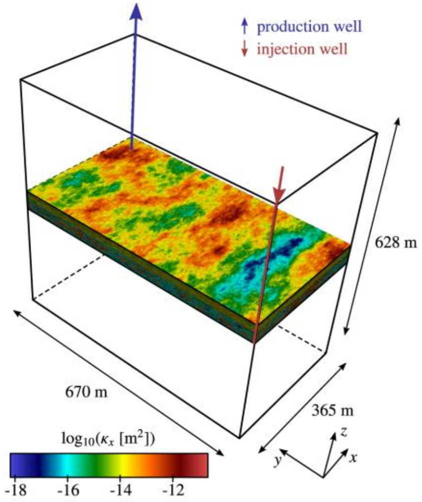
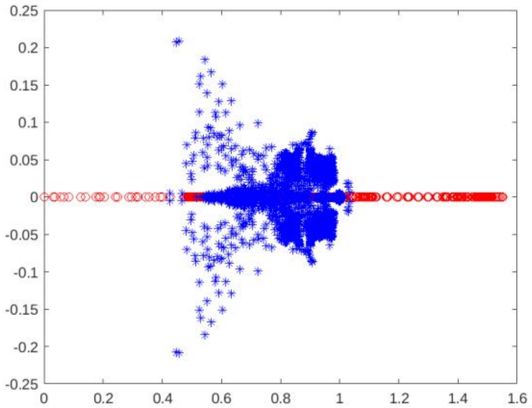
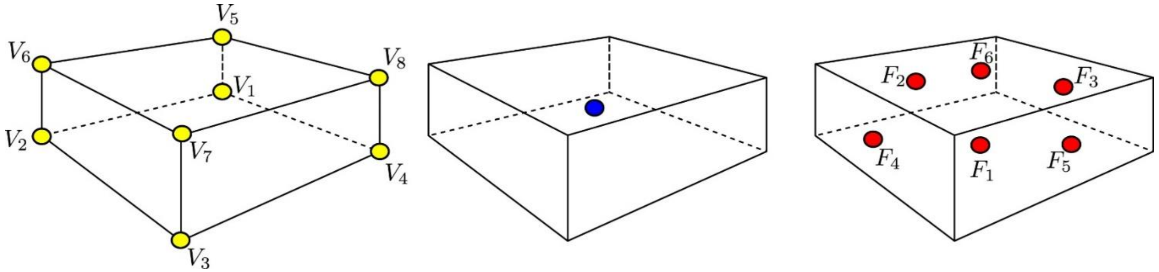
- **Mathematical model**
  - *Mixed hybrid discretization of coupled poromechanics*
- **Linear solver**
  - *Block triangular preconditioning*
  - *Eigenvalue boundedness and robustness*
- **Numerical results**
  - *Bound validation*
  - *Comparison with block diagonal preconditioners*
  - *Computational efficiency*
- **Conclusions and future work**







Mathematical model

Coupled poromechanics (Biot's model)

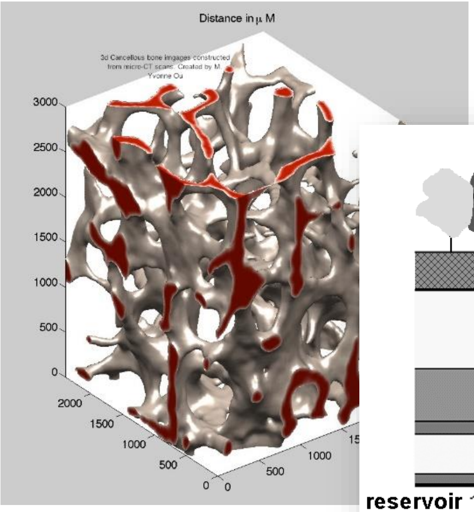
- We focus on coupled elasticity and flow models in porous media written in <u>mixed form</u> (*three-field formulation*):

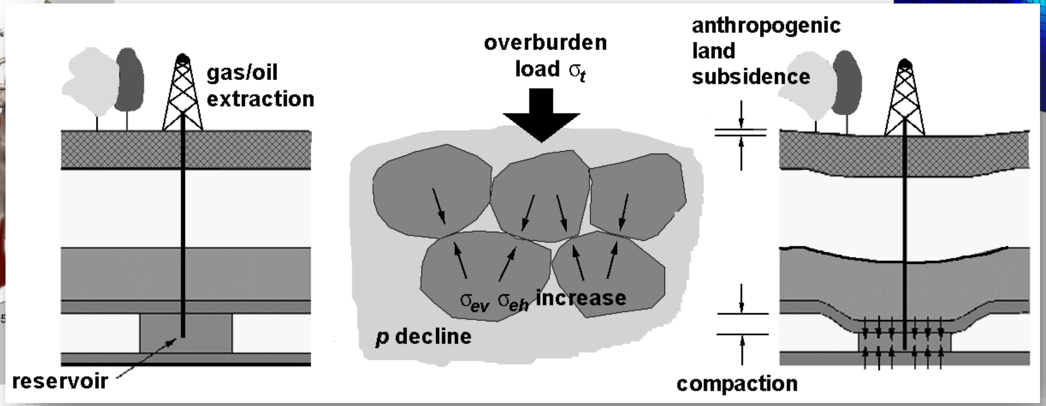
$$\nabla \cdot (\mathbb{C}_{dr} : \nabla^s \boldsymbol{u} - bp\mathbf{1}) = \mathbf{0} \qquad \text{in } \Omega \times (0, t_{max}) \qquad \text{(linear momentum balance)}$$

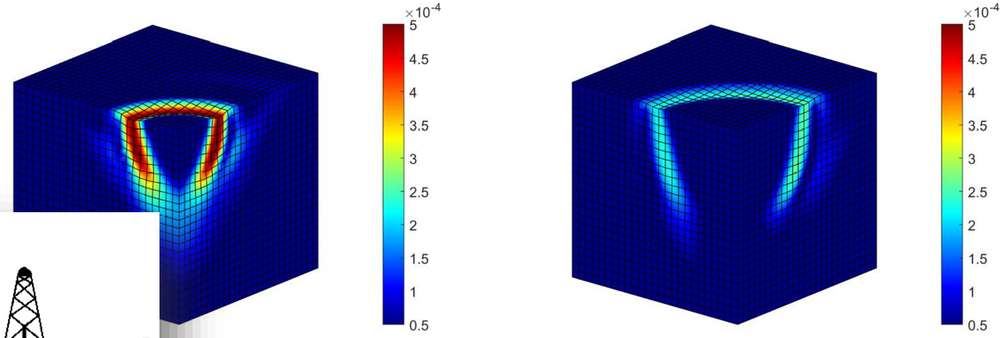
$$\mu \boldsymbol{\kappa}^{-1} \cdot \boldsymbol{q} + \nabla p = \mathbf{0} \qquad \text{in } \Omega \times (0, t_{max}) \qquad \text{(Darcy's law)}$$

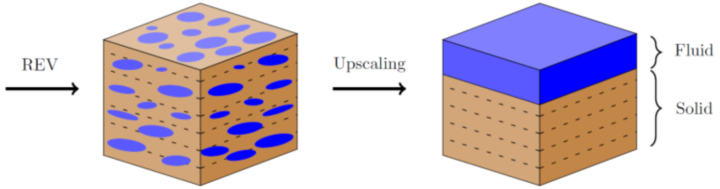
$$b \nabla \cdot \dot{\boldsymbol{u}} + M^{-1} \dot{p} + \nabla \cdot \boldsymbol{q} = f \qquad \text{in } \Omega \times (0, t_{max}) \qquad \text{(fluid mass balance)}$$

- Coupled poromechanical models are widely used in real-world applications, such as environmental, geoscience and biomedical problems





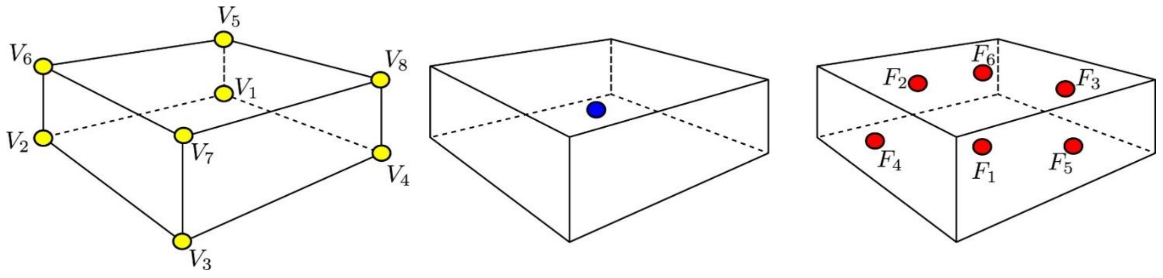




Mathematical model

Mixed hybrid discretization

- A widely used discretization is based on low-order elements, e.g. lowest-order continuous ($\mathbb{Q}_1$), lowest-order Raviart-Thomas ($\mathbb{RT}_0$), piecewise constant ($\mathbb{P}_0$) spaces for displacements, Darcy's velocity and fluid pore pressure, respectively



- Hybridization of the mixed three-field coupled poromechanics problem:
  - One DoF per face per element for the normal component of Darcy's velocity
  - One Lagrange multiplier per face, i.e. interface pressure, to restore the flux continuity

- Attractive features of this formulation: (1) Local (element-wise) mass conservation, (2) Robustness to heterogeneity of material properties, (3) Interesting algebraic properties through static condensation

✓ Weak problem: find $\{\boldsymbol{u}_n^h, \boldsymbol{q}_n^h, p_n^h, \pi_n^h\} \in \boldsymbol{\mathcal{U}}^h \times \boldsymbol{\mathcal{Q}}^h \times \mathcal{P}^h \times \mathcal{B}^h$ for time steps $n = 1, \dots, N$:

$$\left(\nabla^s \boldsymbol{\eta}_i, \mathsf{C}_{dr} : \nabla^s \boldsymbol{u}_n^h\right)_\Omega - \left(\nabla \cdot \boldsymbol{\eta}_i, b p_n^h\right)_\Omega = (\boldsymbol{\eta}_i, \boldsymbol{t}_n)_{\Gamma_\sigma} \qquad \boldsymbol{\eta}_i \in \boldsymbol{\mathcal{U}}^h, i = 1, \dots, n_u$$

$$\left(\boldsymbol{\varphi}_j, \mu \boldsymbol{\kappa}^{-1} \cdot \boldsymbol{q}_n^h\right)_\Omega - \left(\nabla \cdot \boldsymbol{\varphi}_j, p_n^h\right)_\Omega + \left(\boldsymbol{\varphi}_j \cdot \boldsymbol{n}_e, \pi_n^h\right)_{\partial\Omega} = 0 \qquad \boldsymbol{\varphi}_j \in \boldsymbol{\mathcal{Q}}^h, j = 1, \dots, n_q$$

$$\left(\chi_k, b \nabla \cdot \boldsymbol{u}_n^h\right)_\Omega + \Delta t\left(\chi_k, \nabla \cdot \boldsymbol{q}_n^h\right)_\Omega + \left(\chi_k, M^{-1} p_n^h\right)_\Omega = \left(\chi_k, \tilde{f}\right)_\Omega \qquad \chi_k \in \mathcal{P}^h, k = 1, \dots, n_p$$

$$\left(\zeta_l, \boldsymbol{q}_n^h \cdot \boldsymbol{n}_e\right)_{\partial\Omega} = (\zeta_l, \bar{p}_n)_{\Gamma_p} \qquad \zeta_l \in \mathcal{B}^h, l = 1, \dots, n_\pi$$

✓ The mixed-hybrid finite element discretization of coupled poromechanics produces the sequence of linear systems:

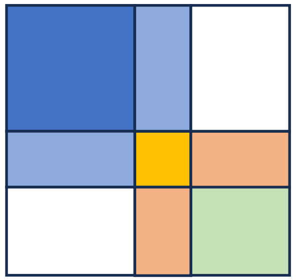
$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{with} \quad \mathbf{A} = \begin{bmatrix} A_{uu} & 0 & A_{up} & 0 \\ 0 & A_{qq} & A_{qp} & A_{q\pi} \\ A_{pu} & \Delta t A_{pq} & A_{pp} & 0 \\ 0 & A_{\pi q} & 0 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \mathbf{q} \\ \mathbf{p} \\ \boldsymbol{\pi} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{f}_u \\ \mathbf{f}_q \\ \mathbf{f}_p \\ \mathbf{f}_\pi \end{bmatrix}$$

✓ For the selected low-order discretization, we have $n_q > n_u > n_\pi > n_p$

Mathematical model

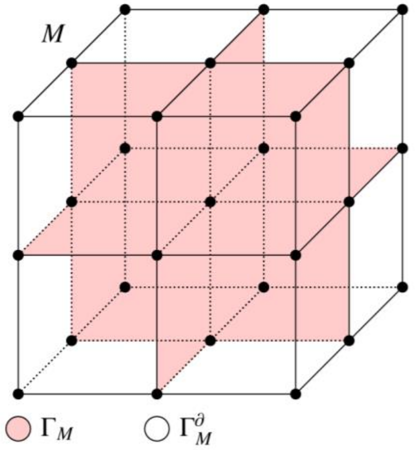
Mixed hybrid discretization

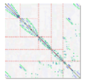
- ✓ Since A_{qq} is block diagonal, the matrix $\mathbf{A}$ can be reduced by <u>static condensation</u>:



$$\mathbf{A} = \begin{bmatrix} A_{uu} & A_{up} & 0 \\ A_{pu} & A_{pp} - \Delta t A_{pq} A_{qq}^{-1} A_{qp} & -\Delta t A_{pq} A_{qq}^{-1} A_{q\pi} \\ 0 & -A_{\pi q} A_{qq}^{-1} A_{qp} & -A_{\pi q} A_{qq}^{-1} A_{q\pi} \end{bmatrix} \quad \Longrightarrow \quad \mathbf{A} = \begin{bmatrix} A_{uu} & A_{up} & 0 \\ A_{pu} & \bar{A}_{pp} & \Delta t A_{p\pi} \\ 0 & A_{\pi p} & A_{\pi\pi} \end{bmatrix}$$

- ✓ The blocks $\bar{A}_{pp}$, $A_{p\pi}$, $A_{\pi p}$ and $A_{\pi\pi}$ are assembled directly element-by-element, with $\bar{A}_{pp}$ a diagonal matrix with non-negative entries and A_{uu} and $A_{\pi\pi}$ symmetric positive definite
- ✓ The selected low-order discretization spaces can be unstable:
  - Undrained conditions ($q \cong 0$) for low permeability ($\kappa \to 0$) or small time step size ($\Delta t \to 0$)
  - Incompressible fluid and solid constituents ($M \to \infty$)
- ✓ We use the <u>pressure-jump stabilization</u> based on the <u>macro-element</u> construction





Mathematical model

Mixed hybrid discretization

- ✓ The stabilization consists of adding the functional $J(\chi_k, p_n^h)$ to the balance equation such that the solvability condition is restored in a macro-element:

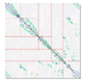
$$
\begin{aligned}
&\left(\nabla^s \boldsymbol{\eta}_i, \mathsf{C}_{dr} : \nabla^s \boldsymbol{u}_n^h\right)_\Omega - \left(\nabla \cdot \boldsymbol{\eta}_i, b p_n^h\right)_\Omega = \left(\boldsymbol{\eta}_i, \boldsymbol{t}_n\right)_{\Gamma_\sigma} && \boldsymbol{\eta}_i \in \boldsymbol{\mathcal{U}}^h, i = 1, \dots, n_u \\
&\left(\chi_k, b \nabla \cdot \boldsymbol{u}_n^h\right)_\Omega + J\left(\chi_k, p_n^h\right) = 0 && \chi_k \in \mathcal{P}^h, k = 1, \dots, n_p
\end{aligned}
$$

- ✓ The functional introduces fictitious fluxes balancing the spurious pressure jumps in adjacent element, weighted by a stabilization parameter β_M automatically selected so as to preserve the non-zero eigenspectrum limits for the local Schur complement:

$$
J\left(\chi_k, p_n^h\right) = \sum_{M \in \mathcal{M}_h} \beta_M |M| \sum_{e \in \Gamma_M} [\![\chi_k]\!]_e [\![p_n^h]\!]_e \qquad \chi_k \in \mathcal{P}^h, k = 1, \dots, n_p
$$

- ✓ The stabilized discrete matrix reads (A_{stab} symmetric positive semidefinite with the stencil of a Laplacian):

$$
\mathbf{A} = \begin{bmatrix} A_{uu} & A_{up} & 0 \\ A_{pu} & \bar{A}_{pp} + A_{stab} & \Delta t A_{p\pi} \\ 0 & A_{\pi p} & A_{\pi\pi} \end{bmatrix}
$$



Linear solver

Block triangular preconditioning

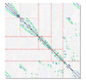
- The robust, efficient and scalable solution to the linear systems arising in large-size real-world applications is a major issue

- The matrix arising from mixed hybrid coupled poromechanics has a double saddle-point structure:

$$\mathbf{A} = \begin{bmatrix} A_{uu} & A_{up} & 0 \\ -A_{pu} & -(\bar{A}_{pp} + A_{stab}) & -\Delta t A_{p\pi} \\ 0 & \Delta t A_{\pi p} & \Delta t A_{\pi\pi} \end{bmatrix} \longrightarrow \boxed{\mathbf{A} = \begin{bmatrix} A & B^T & 0 \\ B & -D & C^T \\ 0 & C & E \end{bmatrix}}$$

- We use Krylov subspace methods (<u>right-preconditioned GMRES</u>) accelerated by a class of <u>block upper triangular preconditioners</u> with the following form:

$$\mathbf{P} = \begin{bmatrix} \hat{A} & B^T & 0 \\ 0 & -\hat{S} & C^T \\ 0 & 0 & \hat{X} \end{bmatrix} \qquad \begin{aligned} &\hat{A} \cong A \\ &\hat{S} \cong \tilde{S} = D + B\hat{A}^{-1}B^T \\ &\hat{X} \cong \tilde{X} = E + C\hat{S}^{-1}C^T \end{aligned}$$



- In order to bound the eigenvalues of $\mathbf{AP}^{-1}$ we consider the scaled eigenproblem:

$$\mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}\mathbf{w} = \lambda\mathbf{D}^{-1/2}\mathbf{P}\mathbf{D}^{-1/2}\mathbf{w} \longrightarrow \begin{bmatrix} \bar{A} & R^T & 0 \\ R & -\bar{D} & K^T \\ 0 & K & \bar{E} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} I & R^T & 0 \\ 0 & -I & K^T \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\bar{A} \cong \hat{A}^{-1/2}A\hat{A}^{-1/2}$ $\quad$ $\bar{D} \cong \hat{S}^{-1/2}D\hat{S}^{-1/2}$ $\quad$ $\bar{E} \cong \hat{X}^{-1/2}E\hat{X}^{-1/2}$ $\quad$ $R \cong \hat{S}^{-1/2}B\hat{X}^{-1/2}$ $\quad$ $K \cong \hat{X}^{-1/2}C\hat{S}^{-1/2}$

- We define the Rayleigh quotient associated to each block as:

$$\gamma_A = \frac{w^T\bar{A}w}{w^Tw} \in \left[\gamma_A^{min}, \gamma_A^{max}\right] \qquad \gamma_S = \frac{w^T\bar{S}w}{w^Tw} \in \left[\gamma_S^{min}, \gamma_S^{max}\right] \qquad \gamma_X = \frac{w^T\bar{X}w}{w^Tw} \in \left[\gamma_X^{min}, \gamma_X^{max}\right]$$

$$\gamma_D = \frac{w^T\bar{D}w}{w^Tw} \in \left[\gamma_D^{min}, \gamma_D^{max}\right] \qquad \gamma_E = \frac{w^T\bar{E}w}{w^Tw} \in \left[\gamma_E^{min}, \gamma_E^{max}\right] \qquad \gamma_R = \frac{w^TRR^Tw}{w^Tw} \in \left[\gamma_R^{min}, \gamma_R^{max}\right]$$

$$\gamma_K = \frac{w^TKK^Tw}{w^Tw} \in \left[\gamma_K^{min}, \gamma_K^{max}\right] \qquad \gamma_S = \gamma_R + \gamma_D \qquad \gamma_X = \gamma_K + \gamma_E$$

Theorem. *If* $0 \le \gamma_D^{min} < 1$, *any complex eigenvalue* λ *of* $\boldsymbol{AP}^{-1}$ *is such that*

$$|\lambda - 1| \le \sqrt{1-\rho}, \qquad \rho = \frac{\gamma_A^{min}\|x\|^2 + \gamma_D^{min}\|y\|^2 + \gamma_E^{min}\|z\|^2}{\|y\|^2 + \gamma_E^{min}\|z\|^2} < 1$$

otherwise all eigenvalues are real.

Proof (sketch). Rewrite the scaled eigenproblem as:

$$x^*\bar{A}x - \lambda\|x\|^2 = (\lambda - 1)x^*R^Ty$$

$$y^*R^Tx - y^*\bar{D}y - (\bar{\lambda} - 1)z^*Ky = -\bar{\lambda}\|y\|^2$$

$$z^*Ky - z^*\bar{E}z = \lambda\|z\|^2$$

and reduce it to a single equation by successive substitutions. Introduce the Rayleigh quotients associated to each block, then separate the real and the imaginary parts assuming that $\lambda = a + ib$. It can be observed that $b \neq 0$ only if $\gamma_D^{min} < 1$ and the modulus of λ is bounded by the thesis. ❑

Theorem. *If λ is a real eigenvalue of $\boldsymbol{AP}^{-1}$, then the following bound holds:*

$$min\left\{\gamma_E^{min}, \gamma_A^{min}, \frac{\gamma_R^{min}}{\gamma_A^{max} + \gamma_R^{min} + \gamma_D^{max}}\right\} \leq \lambda \leq \gamma_A^{max} + \gamma_S^{max} + \gamma_X^{max}$$

Proof (sketch). Compute x and y from the first two equations of the scaled eigenproblem, then using them in the third one yields:

$$[(\lambda - 1)KZ^{-1}(\lambda)K^T + \bar{E} - \lambda I]z = 0$$

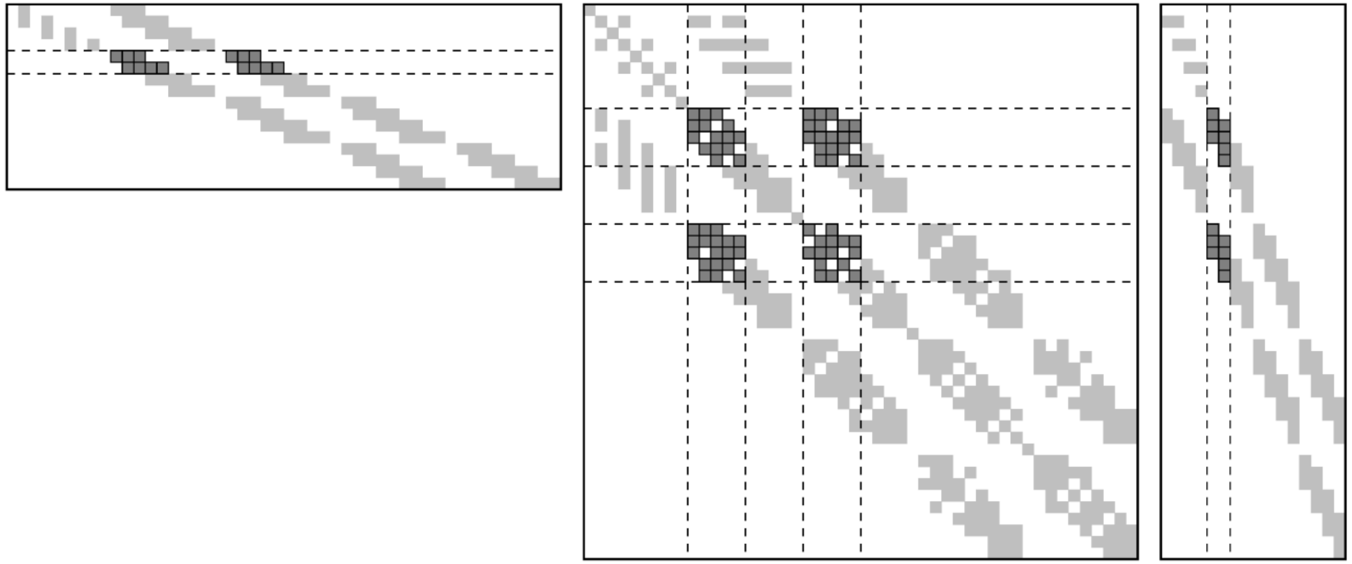
where $Z(\lambda)$ is a particular symmetric positive definite matrix. Introduce the Rayleigh quotients and use the properties of $Z(\lambda)$ to obtain the third-degree polynomial in λ:

$$\lambda^3 - (\gamma_A + \gamma_S + \gamma_X)\lambda^2 - (\gamma_A\gamma_X + \gamma_K + \gamma_E\gamma_S + \gamma_D\gamma_A + \gamma_R)\lambda + \gamma_A\gamma_K + \gamma_E\gamma_A\gamma_D + \gamma_E\gamma_R = 0$$

whose roots are real, positive and bounded by the thesis. ❑

Linear solver

Schur complement approximation



- For the first Schur complement $\tilde{S} \cong D + B\hat{A}^{-1}B^T$ we use a lumped approximation for the contribution $B\hat{A}^{-1}B^T$ taken from the algebraic interpretation of the “fixed-stress” matrix in coupled flow-deformation problems
- The matrix D can be also lumped being the sum of a diagonal matrix and a Laplacian
- Consider a set of m adjacent rows $B^{(i)}$ and adjacent columns $B^{T,(i)}$, the i-th $m \times m$ diagonal block of $\left(\widetilde{B\hat{A}^{-1}B^T}\right)$ reads:

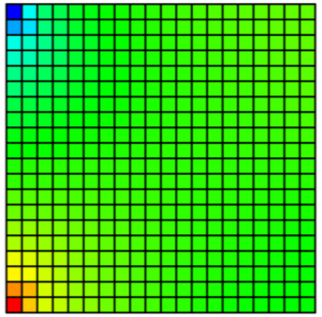
  $D^{(i)} = r\left(B^{(i)}\right)A|_i^{-1}r\left(B^{T,(i)}\right)$

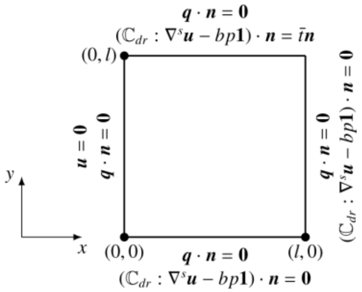
- The second Schur complement $\tilde{X} \cong E + C\hat{S}^{-1}C^T$ we simply replace $\hat{S}$ with its diagonal
- The application of $\tilde{A}^{-1}$ is carried out by an inner AMG, while either Jacobi, IC or AMG is used for $\tilde{S}^{-1}$ and $\tilde{X}^{-1}$

Numerical results

Bound validation

- ✓ 2D cantilever beam problem with $n_u = 3362$, $n_p = 1600$, $n_\pi = 3200$



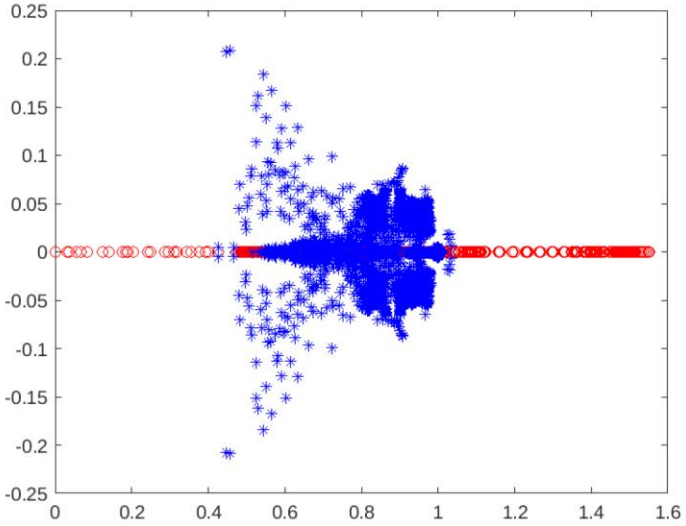


Quantity	Value	Unit
Young's modulus (E)	1×10^5	[Pa]
Poisson's ratio (ν)	0.4	[-]
Biot's coefficient (b)	1.0	[-]
Constrained specific storage (S_ϵ)	0	[Pa]
Isotropic permeability (κ)	1×10^{-7}	[m^2]
Fluid viscosity (μ)	1×10^{-3}	[Pa · s]
Domain size x-y (l)	1.0	[m]

- ✓ The intervals for the coefficients γ are:

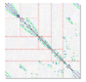
$\gamma_A \in [5.0e-5, 1.0346]$, $\gamma_S \in [0.3665, 1.5582]$,

$\gamma_R \in [4.4e-3, 0.9411]$, $\gamma_D \in [3.8e-3, 0.7512]$, $\gamma_X \in [\,0.5007, 1.5122]$, $\gamma_K \in [0, 0.0112]$, $\gamma_X \in [\,0.5007, 1.5046]$



Triangular preconditioner: bounds	5.01e-05	4.02
Triangular preconditioner: true eigenvalues	5.01e-05	1.55

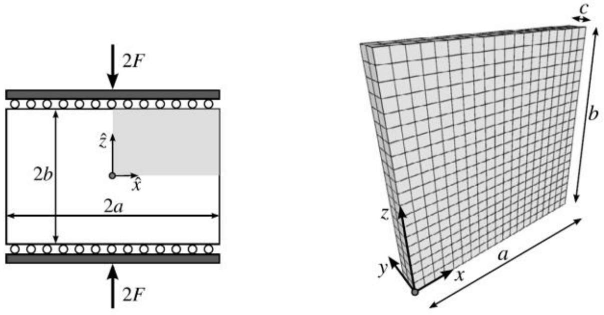
- ✓ The theoretical bounds appear to be sufficiently reliable, especially for the minimum eigenvalue of the preconditioned matrix
- ✓ It is confirmed that the complex eigenvalues lie in a circle centered in 1 with radius smaller than 1



Numerical results

Bound validation

- ✓ 3D Mandel's problem with $n_u = 3969$, $n_p = 800$, $n_\pi = 2880$
- ✓ The intervals for the coefficients γ are:

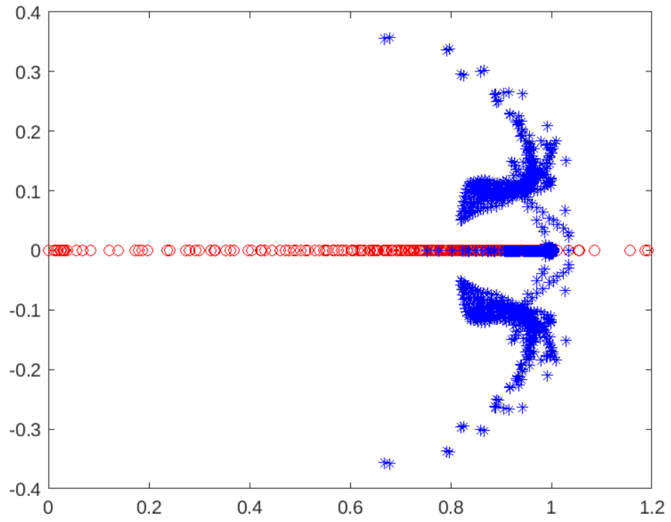


(a) (b)

Symbol	Quantity	Value	Unit
E	Young's modulus	1×10^6	[Pa]
ν	Poisson's ratio	0.2	[-]
b	Biot's coefficient	1.0	[-]
M	Biot's modulus	∞	[Pa]
κ	Isotropic permeability	1×10^{-12}	[m^2]
μ	Fluid viscosity	1×10^{-3}	[Pa·s]
a	Domain size in x-direction	1.0	[m]
b	Domain size in z-direction	1.0	[m]
c	Domain size in y-direction	0.1	[m]
$\lvert F \rvert$	Applied force magnitude	2×10^2	[N·m^{-1}]
t_c	Characteristic consolidation time	900	[s]
τ	Solver tolerance	1×10^{-8}	[-]

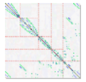
$\gamma_A \in [5.0e-5, 1.0346]$, $\quad \gamma_S \in [0.0100, 1.2132]$,

$\gamma_R \in [7.8e-5, 1.2033]$, $\quad \gamma_D \in [8.0e-5, 0.0100]$, $\quad \gamma_X \in [\,0.9999, 1.0020]$, $\quad \gamma_K \in [0, 0.0021]$, $\quad \gamma_X \in [\,0.9998, 1.0000]$



Triangular preconditioner: bounds	5.00978e-05	3.24985
Triangular preconditioner: true eigenvalues	5.01130e-05	1.19223

- ✓ As before, the bounds are verified also in a 3D problem with a sufficiently tight outcome



Numerical results

Comparison with block diagonal preconditioners

✓ A block triangular preconditioner requires the use of GMRES as a solver, while MINRES could be used with a block diagonal option:

$$\mathbf{P}_D = \begin{bmatrix} \hat{A} & 0 & 0 \\ 0 & \hat{S} & 0 \\ 0 & 0 & \hat{X} \end{bmatrix} \qquad \begin{aligned} &\hat{A} \cong A \\ &\hat{S} \cong \tilde{S} = D + B\hat{A}^{-1}B^T \\ &\hat{X} \cong \tilde{X} = E + C\hat{S}^{-1}C^T \end{aligned}$$

2D Cantilever beam

Triangular preconditioner: bounds			5.01e-05	4.02
Triangular preconditioner: true eigenvalues			5.01e-05	1.55
Diagonal preconditioner: bounds	-1.727	-0.3542	5.0098e-05	3.5151
Diagonal preconditioner: true eigenvalues	-1.125	-0.5546	5.0105e-05	0.9994

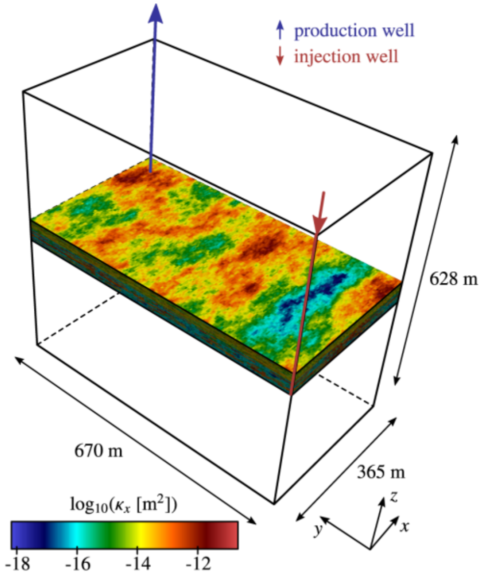
solver(prec)	its	CPU	solver(prec)	its	CPU
GMRES ($\mathcal{P}$)	70	1.379	MINRES ($\mathcal{P}_D$)	185	3.266

3D Mandel's problem

Triangular preconditioner: bounds			5.00978e-05	3.24985
Triangular preconditioner: true eigenvalues			5.01130e-05	1.19223
Diagonal preconditioner: bounds	-1.34472	-0.00969	5.00978e-05	3.13237
Diagonal preconditioner: true eigenvalues	-0.71943	-0.01012	5.01044e-05	1.69465

solver(prec)	its	CPU	solver(prec)	its	CPU
GMRES ($\mathcal{P}$)	98	2.295	MINRES ($\mathcal{P}_D$)	276	6.273

- Augmented SPE-10 benchmark
- 288-m thick layers added on top and bottom to the original setup
- The grid totals 3,410,693 nodes, 10,062,960 faces and 3,326,400 cells, global size of 23,621,439 dofs
- Time-step size of 0.1 days



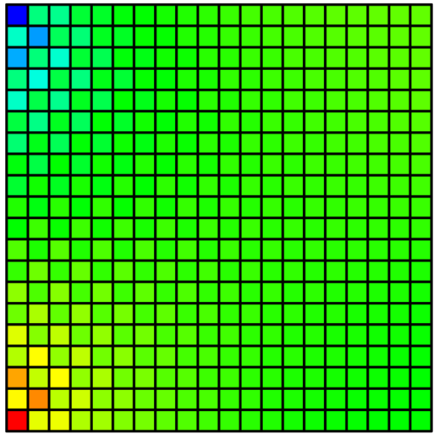
Quantity	Value	Unit
Young's modulus (E)	5×10^9	[Pa]
Poisson's ratio (ν)	0.25	[-]
Biot's coefficient (b)	1.0	[-]
Constrained specific storage (S_ϵ)	0	[Pa]
Reservoir permeability ($\boldsymbol{\kappa}$)	SPE10 data [40]	[m^2]
Overburden/underburden isotropic permeability (κ)	1×10^{-17}	[m^2]
Fluid viscosity (μ)	3×10^{-3}	[Pa · s]
Domain size x	365	[m]
Domain size y	670	[m]
Domain size z	628	[m]

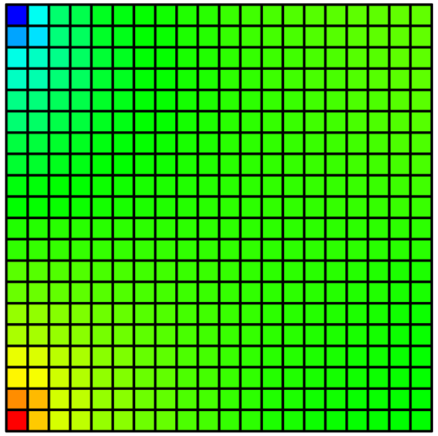
# proc.	# dofs / proc.	# iter.	T_p [s]	T_s [s]	T_t [s]	Efficiency
36	656,151	85	9.8	97.3	107.1	100%
72	328,075	85	5.3	47.6	53.0	101%
144	164,037	88	3.1	22.7	25.8	104%
288	82,018	89	1.8	10.4	12.2	110%
576	41,009	90	1.3	5.3	6.6	101%

Numerical results

Computational efficiency

- ✓ <u>3D cantilever beam</u>
- ✓ Weak scalability analysis
- ✓ GAMG for $\tilde{A}$, diagonal fixed-stress approximation for $\tilde{S}$, Boomer AMG for $\tilde{X}$
- ✓ Different time steps





				Mixed FE ($\mathbf{M}_{II}^{(m)}$)				Hybrid FE ($\mathbf{M}_{II}^{(h)}$)				Hybrid FE ($\mathbf{M}_{III}^{(h)}$)			
	$1/h$	# dofs	# proc.	# iter.	T_p [s]	T_s [s]	T_t [s]	# iter.	T_p [s]	T_s [s]	T_t [s]	# iter.	T_p [s]	T_s [s]	T_t [s]
$\Delta t = 10^{-1}$ s	64	1,884,739	36	55	1.0	2.9	3.9	53	1.0	2.5	3.5	36	3.4	4.8	8.2
	128	14,877,827	288	63	1.5	4.2	5.7	63	1.6	3.6	5.2	47	4.2	6.5	10.7
	256	118,229,251	2268	90	2.6	8.5	11.1	86	2.2	7.7	9.9	63	6.9	9.9	16.8
$\Delta t = 10^{-5}$ s	64	1,884,739	36	56	1.0	3.2	4.2	52	1.0	2.7	3.7	31	3.4	4.3	7.7
	128	14,877,827	288	71	1.3	5.3	6.6	63	1.5	4.1	5.6	43	3.5	6.4	9.9
	256	118,229,251	2268	95	2.8	12.3	15.1	83	2.2	7.9	10.1	59	5.9	10.1	16.0

Separate Displacement Component

Rigid Body Modes

Conclusions

Take-home message and future work

- ✓ **Work objective:** development and analysis of block triangular preconditioners for the double saddle-point problems arising from a mixed hybrid discretization of coupled poromechanics
- ✓ **Main results:**
  1. Theoretical analysis of the eigenvalue distribution of $\mathbf{A}\mathbf{P}^{-1}$, where the double saddle-point problem has all non-zero diagonal blocks
  2. The bounds prove sufficiently tight, with complex eigenvalues all lying in a circle centered at 1 with unitary radius
  3. Block triangular preconditioners with GMRES generally outperform block diagonal ones with MINRES
  4. The proposed implementation appears to be robust and efficient for the solution to large-size real-world problems in coupled poromechanics

Future work

- ✓ Generalizing the method for other applications

Thanks for attending!

Questions?

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