Overlapping Schwarz Preconditioner with Geneo Coarse Space for Nonlocal Equations

Christian Glusa [1] Pierre Marchand [2]

June 11, 2024 — Preconditioning 2024

[1]Sandia National Laboratories

[2]Inria

Non-local/fractional problems

Fractional derivatives are interesting alternative tools for modelling. In particular, fractional space-derivatives

- have been used to model anomalous diffusion (e.g. porous media),
- and, attract interests from the mathematical community.[1][2][3][4]

Focus of this talk: Parallel solver for the Fractional Laplacian.

[1] Bonito, Borthagaray, Nochetto, Otárola, and Salgado 2018.
[2] Čiegis, Starikovičius, Margenov, and Kriauzienė 2017.
[3] Duo, Wang, and Zhang 2019.
[4] Lischke, Pang, Gulian, Song, Glusa, Zheng, Mao, Cai, Meerschaert, Ainsworth, and Karniadakis 2020.

What is the Fractional Laplacian?

- In $\mathbb{R}^d$, for $0 < s < 1$, a definition is

$$(-\Delta)^s u(\mathbf{x}) = C(d,s)\,\mathrm{p.v.}\int_{\mathbb{R}^d} \frac{u(\mathbf{x}) - u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\,\mathrm{d}\mathbf{y}$$

but several other equivalent definitions exist[5].

[5] Kwaśnicki 2017.

What is the Fractional Laplacian?

- In $\mathbb{R}^d$, for $0 < s < 1$, a definition is

$$(-\Delta)^s u(\mathbf{x}) = C(d,s)\,\mathrm{p.\,v.}\int_{\mathbb{R}^d} \frac{u(\mathbf{x}) - u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\,\mathrm{d}\mathbf{y}$$

  but several other equivalent definitions exist[5].

- On a bounded domain Ω, several definitions exist, we use: the Dirichlet Fractional Laplacian

$$\begin{cases} (-\Delta)^s u(\mathbf{x}) = f(\mathbf{x}), & \text{for } \mathbf{x} \in \Omega \\ u(\mathbf{x}) = 0, & \text{for } \mathbf{x} \in \mathbb{R}^d \setminus \Omega \end{cases}$$

[5] Kwaśnicki 2017.

Difficulties with the Fractional Laplacian

Weak formulation: Find $u \in \widetilde{H}^s(\Omega)$ such that

$$a(u,v) := \frac{C(d,S)}{2} \iint_{\Omega\times\Omega} \frac{(u(\mathbf{x})-u(\mathbf{y}))(v(\mathbf{x})-v(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|^{d+2s}} \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} + \frac{C(d,S)}{2s} \int_{\Omega}\int_{\partial\Omega} \frac{u(\mathbf{x})v(\mathbf{x})\mathbf{n}\cdot(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}, \quad \forall v \in \widetilde{H}^s(\Omega)$$

Difficulties with finite element approximation

- Entry computation
- Dense linear system
- Conditioning of linear system

Difficulties with the Fractional Laplacian

Weak formulation: Find $u \in \widetilde{H}^s(\Omega)$ such that

$$a(u,v) := \frac{C(d,S)}{2} \iint_{\Omega\times\Omega} \frac{(u(\mathbf{x}) - u(\mathbf{y}))(v(\mathbf{x}) - v(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|^{d+2s}} \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} + \frac{C(d,S)}{2s} \int_{\Omega}\int_{\partial\Omega} \frac{u(\mathbf{x})v(\mathbf{x})\mathbf{n}\cdot(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}, \quad \forall v \in \widetilde{H}^s(\Omega)$$

Difficulties with finite element approximation

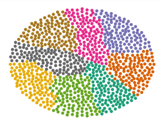
- Entry computation → **Sauter and Schwab quadratures**
- Dense linear system → Compression using **$\mathcal{H}/\mathcal{H}^2$-matrices**
- Conditioning of linear system → condition number **$\sim O(h^{-2s})$**

→ **techniques developed for Boundary integral equations!**[6][7]

[6] Acosta and Borthagaray 2017.
[7] Ainsworth and Glusa 2018.

DDM Preconditioner for non-local problems

Domain decomposition methods (DDM), and in particular, *Schwarz methods*, define preconditioners,



- adapted to parallel computations,
- with some theoretical guarantees.

But they usually require a second level/coarse space.

DDM Preconditioner for non-local problems

Domain decomposition methods (DDM), and in particular, *Schwarz methods*, define preconditioners,

- adapted to **parallel computations**,
- with some **theoretical guarantees**.

For Boundary integral equations

- Coarse space based on coarse meshes[8]
- Adaptive coarse spaces with GenEO[9][10]

→ **Goal**: Adapt the GenEO coarse space to the Fractional Laplacian.

[8] Hebeker 1990; Stephan and Tran 2021.
[9] Spillane, Dolean, Hauret, Nataf, Pechstein, and Scheichl 2014.
[10] Marchand, Claeys, Jolivet, Nataf, and Tournier 2020.

Implementation

- PyNucleus[11]: (developed by C. Glusa)
  - discretization for the fractional Laplacian
  - $\mathcal{H}^2$ compression
- Htool-DDM[12]: (developed P.M. and P.H. Tournier)
  - distributed parallelized operator
  - DDM preconditioners
  - $\mathcal{H}$-matrices
- HPDDM[13]: (developed P. Jolivet)
  - DDM solvers
  - iterative methods

---

[5] https://sandialabs.github.io/PyNucleus/
[6] https://htool-documentation.readthedocs.io/en/latest/
[7] https://github.com/hpddm/hpddm

Implementation

- PyNucleus[11]: (developed by C. Glusa)
    - discretization for the fractional Laplacian
    - $\mathcal{H}^2$ compression
- Htool-DDM[12]: (developed P.M. and P.H. Tournier)
    - distributed parallelized operator
    - DDM preconditioners
    - $\mathcal{H}$-matrices
- HPDDM[13]: (developed P. Jolivet)
    - DDM solvers
    - iterative methods

[5] https://sandialabs.github.io/PyNucleus/
[6] https://htool-documentation.readthedocs.io/en/latest/
[7] https://github.com/hpddm/hpddm

Implementation

- PyNucleus[11]: (developed by C. Glusa)
  - discretization for the fractional Laplacian
  - $\mathcal{H}^2$ compression
- Htool-DDM[12]: (developed P.M. and P.H. Tournier)
  - distributed parallelized operator
  - DDM preconditioners
  - $\mathcal{H}$-matrices
- HPDDM[13]: (developed P. Jolivet)
  - DDM solvers
  - iterative methods

---

[5] https://sandialabs.github.io/PyNucleus/
[6] https://htool-documentation.readthedocs.io/en/latest/
[7] https://github.com/hpddm/hpddm

Table of contents

Dirichlet integral Fractional Laplacian

Function spaces

Geometry

- $\Omega \subset \mathbb{R}^d$ for $d = 2$ or $d = 3$, Lipschitz domain

Fractional Sobolev spaces: For Ω bounded and $\Lambda \subseteq \mathbb{R}^d$ and $0 < s < 1$:

- $H^s(\Lambda) := \{u \in L^2(\Lambda) \,|\, |u|_{H^s(\Lambda)} < \infty\}$ where $|u|_{H^s(\Lambda)} := (u, u)^{1/2}_{H^s(\Lambda)}$ and

$$(u, v)_{H^s(\Lambda)} := \frac{C(d, s)}{2} \iint_{\Lambda \times \Lambda} \frac{(u(\mathbf{x}) - u(\mathbf{y}))(v(\mathbf{x}) - v(\mathbf{y}))}{|\mathbf{x} - \mathbf{y}|^{d+2s}} \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}$$

Function spaces

Geometry

- $\Omega \subset \mathbb{R}^d$ for $d = 2$ or $d = 3$, Lipschitz domain

Fractional Sobolev spaces: For Ω bounded and $\Lambda \subseteq \mathbb{R}^d$ and $0 < s < 1$:

- $H^s(\Lambda) := \{u \in L^2(\Lambda) \,|\, |u|_{H^s(\Lambda)} < \infty\}$ where $|u|_{H^s(\Lambda)} := (u,u)^{1/2}_{H^s(\Lambda)}$ and

$$(u,v)_{H^s(\Lambda)} := \frac{C(d,s)}{2} \iint_{\Lambda\times\Lambda} \frac{(u(\mathbf{x}) - u(\mathbf{y}))(v(\mathbf{x}) - v(\mathbf{y}))}{|\mathbf{x} - \mathbf{y}|^{d+2s}} \, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}$$

- $\widetilde{H}^s(\Omega) := \{u \in H^s(\mathbb{R}^d) \,|\, \mathrm{supp}(u) \subset \overline{\Omega}\}$ equipped with the norm

$$\|\varphi\|^2_{\widetilde{H}^s(\Omega)} := (\varphi, \varphi)_{H^s(\mathbb{R}^d)}.$$

Weak formulation

For $u, v \in \widetilde{H}^s(\Omega)$, we have

$$\begin{aligned}
\langle(-\Delta)^s u, v\rangle &= C(d,s) \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(u(\mathsf{x}) - u(\mathsf{y}))v(\mathsf{x})}{|\mathsf{x}-\mathsf{y}|^{d+2s}}\,\mathrm{d}\mathsf{x}\,\mathrm{d}\mathsf{y} \\
&= \frac{C(d,s)}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(u(\mathsf{x}) - u(\mathsf{y}))(v(\mathsf{x}) - v(\mathsf{y}))}{|\mathsf{x}-\mathsf{y}|^{d+2s}}\,\mathrm{d}\mathsf{x}\,\mathrm{d}\mathsf{y} \\
&= (u, v)_{H^s(\mathbb{R}^d)}, \quad \forall u, v \in \widetilde{H}^s(\Omega)
\end{aligned}$$

Weak formulation

Variational formulation: Find $u \in \widetilde{H}^s(\Omega)$ such that

$$(u, v)_{H^s(\mathbb{R}^d)} = \langle f, v \rangle, \quad \forall v \in \widetilde{H}^s(\Omega).$$

Weak formulation

Variational formulation: Find $u \in \widetilde{H}^s(\Omega)$ such that

$$(u, v)_{H^s(\mathbb{R}^d)} = \langle f, v \rangle, \quad \forall v \in \widetilde{H}^s(\Omega).$$

Finite element discretization: Find $u_h \in V_h \subset \widetilde{H}^s(\Omega)$ such that

$$(u_h, v_h)_{H^s(\mathbb{R}^d)} = \langle f, v_h \rangle, \quad \forall v_h \in V_h,$$

where V_h can be the usual P^1 Lagrange finite elements.

Weak formulation

Variational formulation: Find $u \in \widetilde{H}^s(\Omega)$ such that

$$(u, v)_{H^s(\mathbb{R}^d)} = \langle f, v \rangle, \quad \forall v \in \widetilde{H}^s(\Omega).$$

Finite element discretization: Find $u_h \in V_h \subset \widetilde{H}^s(\Omega)$ such that

$$(u_h, v_h)_{H^s(\mathbb{R}^d)} = \langle f, v_h \rangle, \quad \forall v_h \in V_h,$$

where V_h can be the usual P^1 Lagrange finite elements.

Remark

$$(u_h, v_h)_{H^s(\mathbb{R}^d)} = (u_h, v_h)_{H^s(\Omega)} + C(d, s) \int_\Omega \int_{\mathbb{R}^d \setminus \overline{\Omega}} \frac{u(\mathbf{x}) v(\mathbf{x})}{|\mathbf{x} - \mathbf{y}|^{d+2s}} \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y}$$

Weak formulation

Variational formulation: Find $u \in \widetilde{H}^s(\Omega)$ such that

$$(u, v)_{H^s(\mathbb{R}^d)} = \langle f, v \rangle, \quad \forall v \in \widetilde{H}^s(\Omega).$$

Finite element discretization: Find $u_h \in V_h \subset \widetilde{H}^s(\Omega)$ such that

$$(u_h, v_h)_{H^s(\mathbb{R}^d)} = \langle f, v_h \rangle, \quad \forall v_h \in V_h,$$

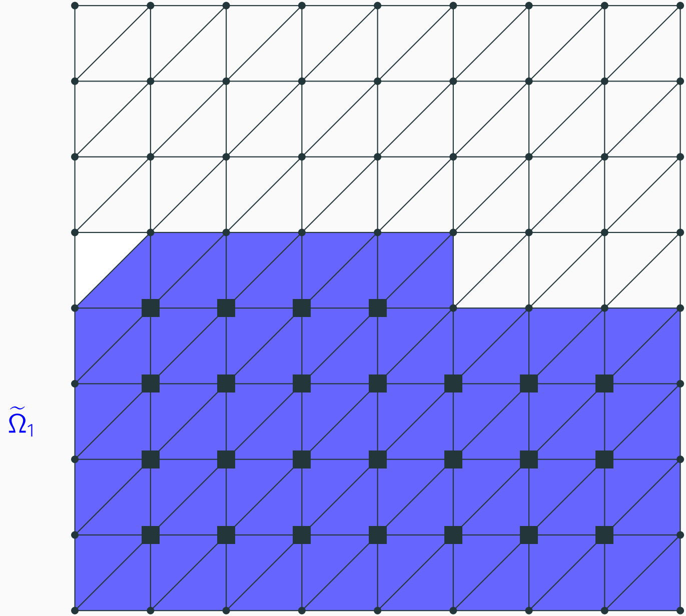
where V_h can be the usual P^1 Lagrange finite elements.

Remark

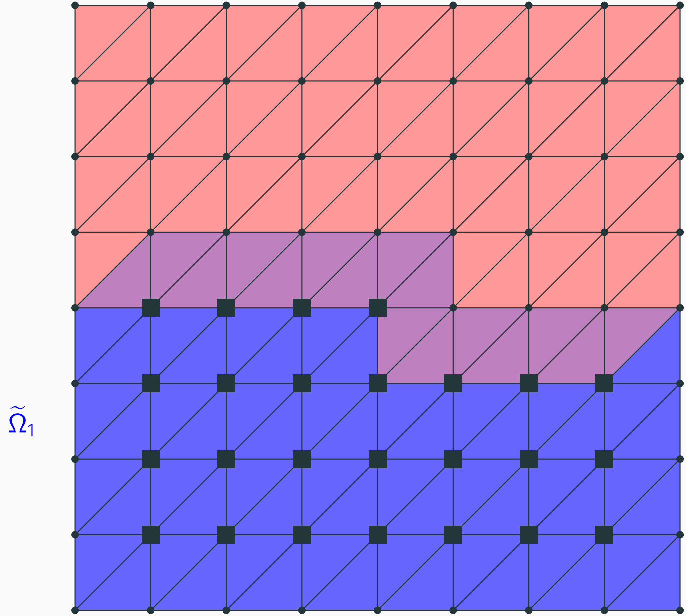
$$(u_h, v_h)_{H^s(\mathbb{R}^d)} = (u_h, v_h)_{H^s(\Omega)} + \frac{C(d,s)}{2s} \int_\Omega \int_{\partial\Omega} \frac{u(\mathsf{x}) v(\mathsf{x}) \mathsf{n}_\mathsf{y} \cdot (\mathsf{x} - \mathsf{y})}{|\mathsf{x} - \mathsf{y}|^{d+2s}} \, \mathrm{d}\mathsf{x} \, \mathrm{d}\mathsf{y}$$

Domain decomposition

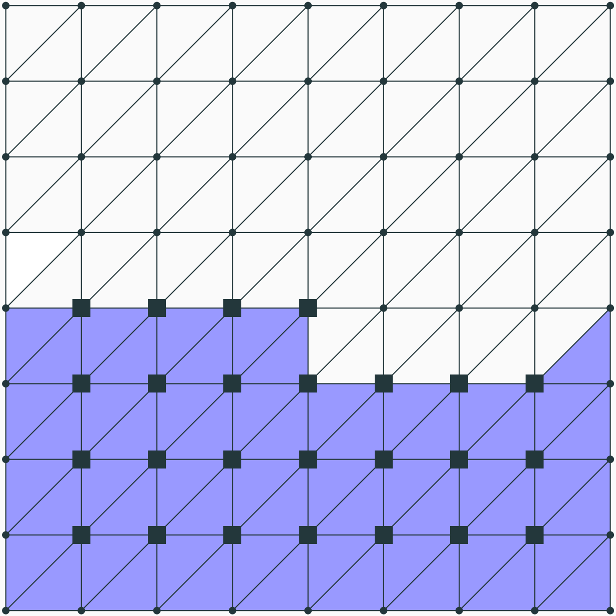
Example of decomposition



Example of decomposition



Example of decomposition



$\Omega_1 \subset \widetilde{\Omega}_1$

Notations

Subdomains

- $\mathsf{dof}_{h,p} \subset \{1, \ldots, N\}$
- $\mathcal{V}_{h,p} := \mathsf{Span}(\varphi_j|_{\widetilde{\Omega}_p} \,|\, j \in \mathsf{dof}_{h,p}) \subset \widetilde{H}^s(\widetilde{\Omega}_p)$,
- $\Omega_p := \widetilde{\Omega}_p \setminus \cup_{j \notin \mathsf{dof}_{h,p}} \mathsf{supp}(\varphi_\mathsf{j}) \subset \widetilde{\Omega}_p$

Decomposition

- Extension by zero: $\mathsf{R}_p^T \in \mathbb{R}^{N \times N_p}$,
- Restriction matrices: R_p
- Partition of unity: diagonal matrices $\mathsf{D}_p \in \mathbb{R}^{N_p \times N_p}$ s.t.

$$\sum_{p=1}^{n} \mathsf{R}_p^T \mathsf{D}_p \mathsf{R}_p = \mathsf{I}_d.$$

Additive Schwarz preconditioner

Additive Schwarz Preconditioner

$$\mathsf{P}_{\mathrm{ASM}} = \mathsf{R}_0^T(\mathsf{R}_0\mathsf{A}_h\mathsf{R}_0^T)^{-1}\mathsf{R}_0 + \sum_{p=1}^{n} \mathsf{R}_p^T(\mathsf{R}_p\mathsf{A}_h\mathsf{R}_p^T)^{-1}\mathsf{R}_p$$

- $\mathsf{Z} = \mathsf{R}_0^T \in \mathbb{R}^{N\times N_0}$, an interpolation operator from the *coarse space* to the finite element space
- The coarse space $\mathcal{V}_{h,0}$ is spanned by the columns of Z

Hypotheses of the Fictitious Space lemma[14]

(H1) $\|\sum_{p=0}^{n} \mathsf{R}_p^T \mathbf{u}_h^p\|_{\mathsf{A}_h}^2 \leq c_R \sum_{p=0}^{n} \|\mathsf{R}_p^T \mathbf{u}_h^p\|_{\mathsf{A}_h}^2 \quad \forall (\mathbf{u}_h^p)_{p=0}^n \in \prod_{p=0}^{n} \mathbb{R}^{N_p}$,

(H2) For $\mathbf{u}_h \in V_h$, how can we define $(\mathbf{u}_h^p)_{p=0}^n \in \prod_{p=0}^{n} \mathbb{R}^{N_p}$ s.t. $\mathbf{u}_h = \sum_{p=0}^{n} \mathsf{R}_p^T \mathbf{u}_h^p$ and

$$c_T \sum_{p=0}^{n} \|\mathsf{R}_p^T \mathbf{u}_h^p\|_{\mathsf{A}_h}^2 \leq \|\mathbf{u}_h\|_{\mathsf{A}_h}^2,$$

Result

$$\mathrm{cond}_2(\mathsf{P}_{\mathrm{ASM}} \mathsf{A}_h) \leq \frac{c_R}{c_T}.$$

[14] Nepomnyaschikh 1992.

Coarse space: GenEO approach

Pragmatic approach: For $\mathsf{u}_h = \sum_{p=0}^{n} \mathsf{R}_p^T \mathsf{u}_h^p$ and some B_p,

$$\sum_{p=0}^{n} \|\mathsf{R}_p^T \mathsf{u}_h^p\|_{\mathsf{A}_h}^2 \leq 2\|\mathsf{u}_h\|_{\mathsf{A}_h}^2 + (1+5N_c) \sum_{p=1}^{n} \|\mathsf{R}_p^T \mathsf{u}_h^p\|_{\mathsf{A}_h}^2$$

$$\sum_{p=1}^{n} (\mathsf{B}_p \mathsf{R}_p \mathsf{u}_h, \mathsf{R}_p \mathsf{u}_h) \overset{?}{\leq} \|u_h\|_{\widetilde{H}^s(\Omega)}^2 = \|\mathsf{u}_h\|_{\mathsf{A}_h}^2$$

Coarse space: GenEO approach

Pragmatic approach: For $\mathbf{u}_h = \sum_{p=0}^{n} \mathsf{R}_p^T \mathbf{u}_h^p$ and some B_p,

$$\sum_{p=0}^{n} \|\mathsf{R}_p^T \mathbf{u}_h^p\|_{\mathsf{A}_h}^2 \leq 2\|\mathbf{u}_h\|_{\mathsf{A}_h}^2 + (1 + 5N_c) \sum_{p=1}^{n} \|\mathsf{R}_p^T \mathbf{u}_h^p\|_{\mathsf{A}_h}^2$$

$$\sum_{p=1}^{n} (\mathsf{B}_p \mathsf{R}_p \mathbf{u}_h, \mathsf{R}_p \mathbf{u}_h) \overset{?}{\leq} \|u_h\|_{\widetilde{H}^s(\Omega)}^2 = \|\mathbf{u}_h\|_{\mathsf{A}_h}^2$$

Coarse space: GenEO approach

Pragmatic approach: For $\mathbf{u}_h = \sum_{p=0}^{n} \mathsf{R}_p^T \mathbf{u}_h^p$ and some B_p,

$$\sum_{p=0}^{n} \|\mathsf{R}_p^T \mathbf{u}_h^p\|_{\mathsf{A}_h}^2 \leq 2\|\mathbf{u}_h\|_{\mathsf{A}_h}^2 + (1 + 5N_c) \sum_{p=1}^{n} \|\mathsf{R}_p^T \mathbf{u}_h^p\|_{\mathsf{A}_h}^2$$

$$\sum_{p=1}^{n} (\mathsf{B}_p \mathsf{R}_p \mathbf{u}_h, \mathsf{R}_p \mathbf{u}_h) \overset{?}{\lesssim} \|u_h\|_{\widetilde{H}^s(\Omega)}^2 = \|\mathbf{u}_h\|_{\mathsf{A}_h}^2$$

GenEO coarse space: A sufficient condition is, for some τ,

$$(\mathsf{A}_h \mathsf{R}_p^T \mathbf{u}_h^p, \mathsf{R}_p^T \mathbf{u}_h^p) \leq \tau (\mathsf{B}_p \mathsf{R}_p \mathbf{u}_h, \mathsf{R}_p \mathbf{u}_h). \tag{1}$$

We introduce the following eigenvalue problem: find $(\lambda_k^p, \mathbf{v}_k^p)$ s.t.

$$\mathsf{D}_p \mathsf{R}_p \mathsf{A}_h \mathsf{R}_p^T \mathsf{D}_p \mathbf{v}_k^p = \lambda_k^p \mathsf{B}_p \mathbf{v}_k^p,$$

to remove the part of $\mathsf{R}_p \mathbf{u}_h$ that does not satisfy (1) to define $\mathbf{u}_h^p$.

Localization inequalities

Continuous injection: B_p, mass matrix on Ω_p

$$\sum_{p=1}^{n} \|u_h|_{\Omega_p}\|^2_{L^2(\Omega_p)} \lesssim \|u_h\|^2_{L^2(\Omega)} \lesssim \|u_h\|^2_{\widetilde{H}^s(\Omega)},$$

Inverse inequality: B_p, rigidity matrix on Ω_p

$$\sum_{p=1}^{n} h^{1-s} \|\nabla u|_{\Omega_p}\|^2_{L^2(\Omega_p)} \lesssim h^{1-s} \|\nabla u_h\|^2_{L^2(\Omega)} \lesssim \|u_h\|^2_{\widetilde{H}^s(\Omega)}.$$

H^s-localization:

$$\sum_{p=1}^{n} (u_h|_{\Omega_p}, u_h|_{\Omega_p})_{H^s(\Omega)} \leq k_1 \|u_h\|^2_{\widetilde{H}^s(\Omega)}.$$

Localization inequalities

Continuous injection: B_p, mass matrix on Ω_p

$$\sum_{p=1}^{n} \|u_h|_{\Omega_p}\|^2_{L^2(\Omega_p)} \lesssim \|u_h\|^2_{L^2(\Omega)} \lesssim \|u_h\|^2_{\widetilde{H}^s(\Omega)},$$

Inverse inequality: B_p, rigidity matrix on Ω_p

$$\sum_{p=1}^{n} h^{1-s} \|\nabla u|_{\Omega_p}\|^2_{L^2(\Omega_p)} \lesssim h^{1-s} \|\nabla u_h\|^2_{L^2(\Omega)} \lesssim \|u_h\|^2_{\widetilde{H}^s(\Omega)}.$$

H^s-localization:

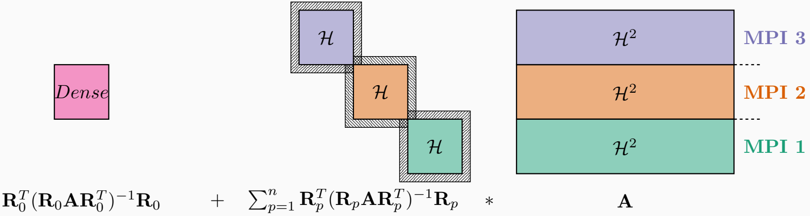
$$\sum_{p=1}^{n} (u_h|_{\Omega_p}, u_h|_{\Omega_p})_{H^s(\Omega)} \leq k_1 \|u_h\|^2_{\widetilde{H}^s(\Omega)}.$$

$\rightarrow \mathrm{cond}(\mathsf{P}_{\mathrm{ASM}} \mathsf{A}_h) \leq 5N_c \left(2 + k_1 \tau (1 + 5N_c)\right)$

(Preliminary) Numerical experiments

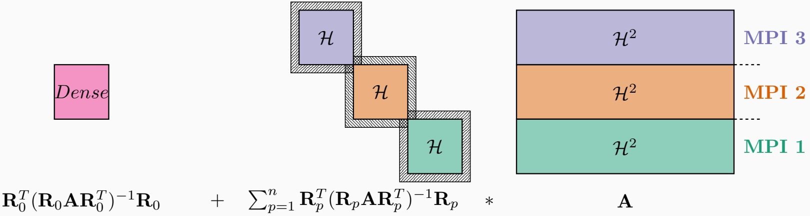
Test case

- Fractional Poisson problem in a disk with $f = 1$.
- Error between interpolated analytical solution and numerical solution $\leq 5\%$
- Computations on IDCS mesocentre.
- Implementation: PyNucleus' $\mathcal{H}^2$-matrices and Htool's $\mathcal{H}$-matrices.



Test case

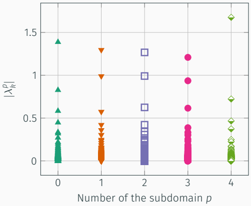
- Fractional Poisson problem in a disk with $f = 1$.
- Error between interpolated analytical solution and numerical solution $\leq 5\%$
- Computations on IDCS mesocentre.
- Implementation: PyNucleus' $\mathcal{H}^2$-matrices and Htool's $\mathcal{H}$-matrices.



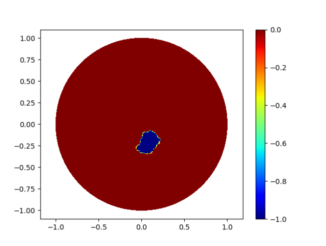
Two experiments

1. Local eigenproblems with 64 100 P^1 elements and $p = 72$.
2. Strong scaling with 258 171 P^1 elements.

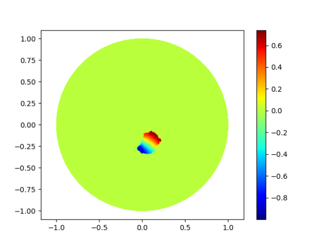
Local eigenproblems: Rigidity matrix for $s = 0.7$



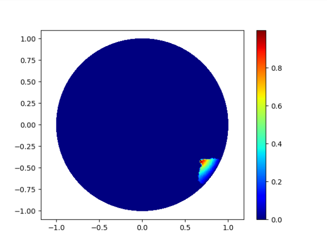
Eigenvalues



First vector for $p = 0$.

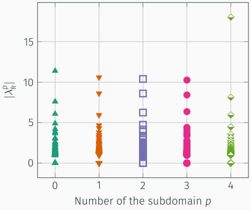


Second vector for $p = 0$.

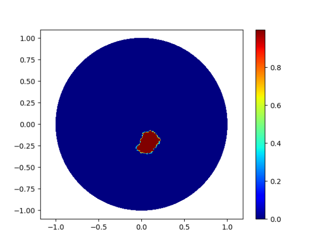


First vector for $p = 4$.

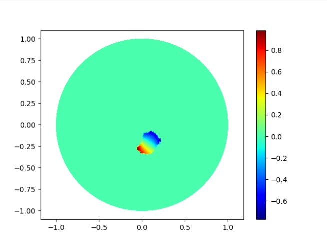
Local eigenproblems: H^s localization for $s = 0.7$



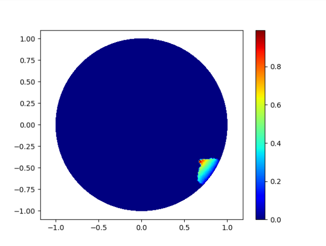
Eigenvalues



First vector for $p = 0$.

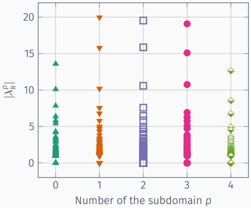


Second vector for $p = 0$.

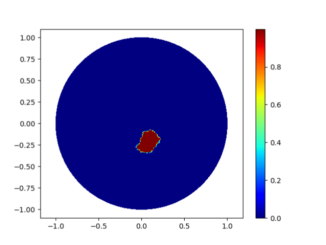


First vector for $p = 4$.

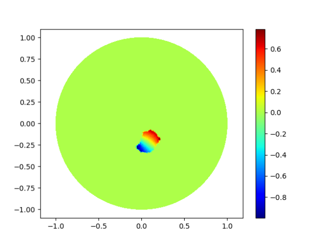
Local eigenproblems: H^s localization for $s = 0.9$



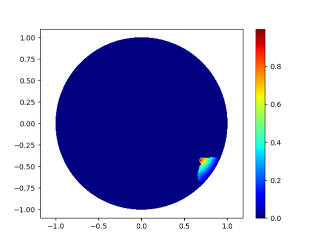
Eigenvalues



First vector for $p = 0$.

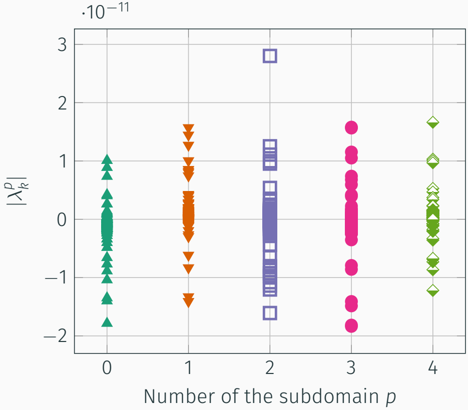


Second vector for $p = 0$.



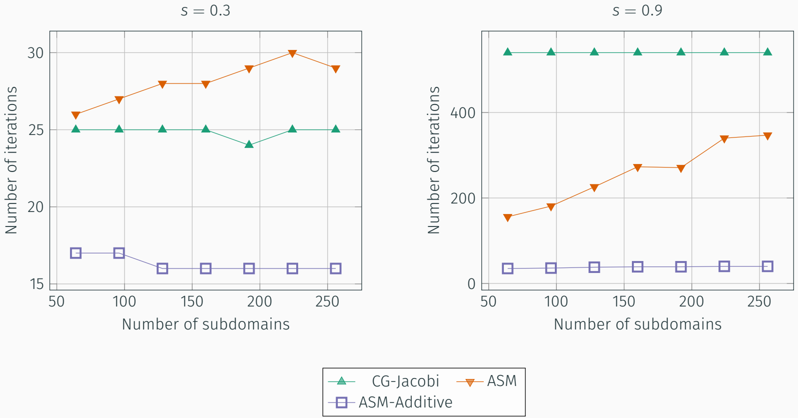
First vector for $p = 4$.

Local eigenproblems: Mass matrix for $s = 0.7$



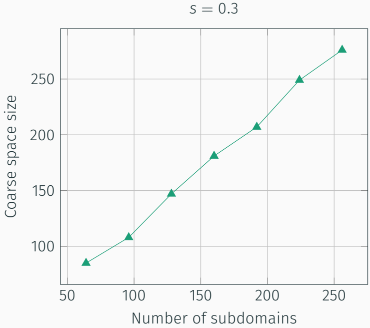
Eigenvalues

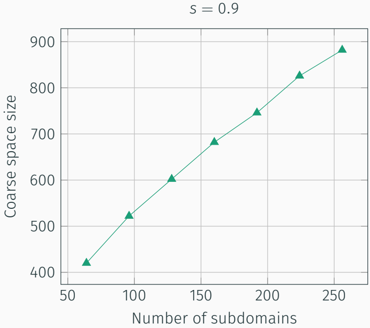
Strong scaling with H^s-localization



Strong scaling in 2D for 258 171 P^1 Lagrange elements and $\tau = 10$.

Strong scaling





Strong scaling in 2D for 258 171 P^1 Lagrange elements and $\tau = 10$.

Conclusion

Summary

What is done:

- We extended GenEO to the Dirichlet integral Fractional Laplacian.
- We checked numerically the relevance of the coarse space.

Summary

What is done:

- We extended GenEO to the Dirichlet integral Fractional Laplacian.
- We checked numerically the relevance of the coarse space.

What remains: Performance

- HCA compression for preconditioner, better handling of the overlap and $\mathcal{H}$-LU factorization, ...
- Try different local eigensolvers (ARPACK, LOBPCG,...).

Summary

What is done:

- We extended GenEO to the Dirichlet integral Fractional Laplacian.
- We checked numerically the relevance of the coarse space.

What remains: Performance

- HCA compression for preconditioner, better handling of the overlap and $\mathcal{H}$-LU factorization, ...
- Try different local eigensolvers (ARPACK, LOBPCG,...).

What could be explored: Non-local GenEO for

- Negative order fractional problem ($s < 0$).
- Interpolation problems (RBF interpolation)
- Non-hermitian problem (acoustic with boundary integral operators)

Summary

What is done:

- We extended GenEO to the Dirichlet integral Fractional Laplacian.
- We checked numerically the relevance of the coarse space.

What remains: Performance

- HCA compression for preconditioner, better handling of the overlap and $\mathcal{H}$-LU factorization, ...
- Try different local eigensolvers (ARPACK, LOBPCG,...).

What could be explored: Non-local GenEO for

- Negative order fractional problem ($s < 0$).
- Interpolation problems (RBF interpolation)
- Non-hermitian problem (acoustic with boundary integral operators)

Thank you for your attention!

References

Bibliography

- Acosta, Gabriel and Juan Pablo Borthagaray (2017). "A fractional Laplace equation: regularity of solutions and Finite Element approximations". In: *SIAM Journal on Numerical Analysis* 55.2, pp. 472–495.
- Ainsworth, Mark and Christian Glusa (2018). "Towards an Efficient Finite Element Method for the Integral Fractional Laplacian on Polygonal Domains". In: *Contemporary Computational Mathematics - A Celebration of the 80th Birthday of Ian Sloan*. Springer International Publishing, pp. 17–57. DOI: 10.1007/978-3-319-72456-0_2.

Bonito, Andrea, Juan Pablo Borthagaray, Ricardo H. Nochetto, Enrique Otárola, and Abner J. Salgado (Mar. 2018). “Numerical methods for fractional diffusion”. In: *Computing and Visualization in Science* 19.5-6, pp. 19–46. DOI: 10.1007/s00791-018-0289-y.

Čiegis, Raimondas, Vadimas Starikovičius, Svetozar Margenov, and Rima Kriauzienė (June 2017). “Parallel solvers for fractional power diffusion problems”. In: *Concurrency and Computation: Practice and Experience* 29.24, e4216. DOI: 10.1002/cpe.4216.

Duo, Siwei, Hong Wang, and Yanzhi Zhang (2019). “A comparative study on nonlocal diffusion operators related to the fractional Laplacian”. In: *Discrete & Continuous Dynamical Systems - B* 24.1, pp. 231–256. DOI: 10.3934/dcdsb.2018110.

Hebeker, Friedrich K. (1990). “On a parallel Schwarz algorithm for symmetric strongly elliptic integral equations”. In: *Domain Decomposition Methods for Partial Differential Equations*. Ed. by Roland Glowinski, Yuri A. Kuznetsov, Gérard Meurant, Jacques Périaux, and Olaf B. Widlund. SIAM, pp. 382–393.

Kwaśnicki, Mateusz (Jan. 2017). “Ten equivalent definitions of the fractional laplace operator”. In: *Fractional Calculus and Applied Analysis* 20.1. DOI: 10.1515/fca-2017-0002.

Lischke, Anna, Guofei Pang, Mamikon Gulian, Fangying Song, Christian Glusa, Xiaoning Zheng, Zhiping Mao, Wei Cai, Mark M. Meerschaert, Mark Ainsworth, and George Em Karniadakis (Mar. 2020). “What is the fractional Laplacian? A comparative review with new results”. In: *Journal of Computational Physics* 404, p. 109009. DOI: 10.1016/j.jcp.2019.109009.

Marchand, Pierre, Xavier Claeys, Pierre Jolivet, Frédéric Nataf, and Pierre-Henri Tournier (Sept. 2020). "Two-level preconditioning for h-version boundary element approximation of hypersingular operator with GenEO". In: *Numerische Mathematik* 146.3, pp. 597–628. DOI: 10.1007/s00211-020-01149-5.

Nepomnyaschikh, Sergey V. (1992). "Decomposition and fictitious domains methods for elliptic boundary value problems". In: *Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations*. Philadelphia, PA: Society for Industrial and Applied Mathematics, pp. 62–72.

Sauter, Stefan A. and Christoph Schwab (2011). *Boundary element methods*. Vol. 39. Springer Series in Computational Mathematics. Translated and expanded from the 2004 German original. Springer-Verlag, Berlin, pp. xviii+561. ISBN: 978-3-540-68092-5. DOI: 10.1007/978-3-540-68093-2.

Spillane, Nicole, Victorita Dolean, Patrice Hauret, Frédéric Nataf, Clemens Pechstein, and Robert Scheichl (2014). “Abstract robust coarse spaces for systems of PDEs via generalized eigenproblems in the overlaps”. In: *Numerische Mathematik* 126.4, pp. 741–770. ISSN: 0945-3245. DOI: 10.1007/s00211-013-0576-y. URL: http://dx.doi.org/10.1007/s00211-013-0576-y.

Stephan, Ernst P. and Thanh Tran (2021). *Schwarz Methods and Multilevel Preconditioners for Boundary Element Methods*. Springer International Publishing. ISBN: 9783030792831. DOI: 10.1007/978-3-030-79283-1.

Appendix

Context

Algebraic system

$$\mathsf{A}_h \mathsf{u}_h = \mathsf{f}, \quad \text{with } \mathsf{u}_h \in \mathbb{R}^d$$

A_h a dense matrix usually compressed

Context

Algebraic system

$$\mathbf{A}_h \mathbf{u}_h = \mathbf{f}, \quad \text{with } \mathbf{u}_h \in \mathbb{R}^d$$

$\mathbf{A}_h$ a dense matrix usually compressed

Solvers

- Direct methods:
  - ⊕ Factorisation can be stored for multi-rhs
  - ⊖ Expensive for dense matrices (complexity in $O(N^3)$)
  - ⊕ Possibility to use $\mathcal{H} - LU$ decomposition
- Iterative methods:
  - ⊕ Less intrusive
  - ⊕ Only matrix-vector products ($O(N^2)$ or quasi linear complexity with compression)
  - ⊖ But ill-conditioned, especially when the mesh is refined

Context

Algebraic system

$$\mathsf{A}_h \mathbf{u}_h = \mathbf{f}, \quad \text{with } \mathbf{u}_h \in \mathbb{R}^d$$

A_h a dense matrix usually compressed

Solvers

- Direct methods:
  - ➕ Factorisation can be stored for multi-rhs
  - ➖ Expensive for dense matrices (complexity in $O(N^3)$)
  - ➕ Possibility to use $\mathcal{H} - LU$ decomposition
- Iterative methods:
  - ➕ Less intrusive
  - ➕ Only matrix-vector products ($O(N^2)$ or quasi linear complexity with compression)
  - ➖ But ill-conditioned, especially when the mesh is refined

$\Longrightarrow$ preconditioning techniques: $\mathsf{P}\mathsf{A}_h \mathbf{u}_h = \mathsf{P}\mathbf{f}$

Splitting norms

Lemma

For $(u_p)_{1\leq p\leq n} \in \prod_{p=1}^{n} \widetilde{H}^s(\widetilde{\Omega}_p)$, *we have the following inequality:*

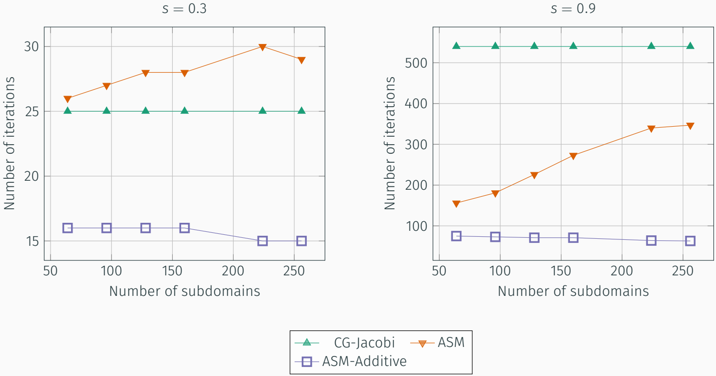
$$\left\| \sum_{p=1}^{n} E_{\widetilde{\Omega}_p}(u_p) \right\|_{\widetilde{H}^s(\Omega)}^2 \leq \frac{5}{2} N_c \sum_{p=1}^{n} \|u_p\|_{\widetilde{H}^s(\widetilde{\Omega}_p)}^2 ,$$

where N_c *is the number of colors.*

Proof:

1. Similar result without overlap (minor adaptation from Sauter and Schwab (2011, Lemma 4.1.49 (b))).
2. Colouring argument.

Strong scaling



Strong scaling in 2D for 258 171 P^1 Lagrange elements with 2 eigenvectors by subdomain.