An Augmented Lagrangian Preconditioner for the Control of the Navier–Stokes equations[1]

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Overview

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Navier–Stokes Control Problems

PDE-Constrained Optimization Problems

A general PDE-Constrained Optimization Problem can be formulated as

$$\min_{v,u} \ \frac{1}{2} \|v - v_d\|^2_{L^2(\mathcal{Q})} + \frac{\beta}{2} \|u\|^2_{L^2(\mathcal{Q})}$$

subject to

$$\mathcal{D}v = u \qquad + \mathrm{BCs}$$

where $\mathcal{D}$ is a differential operator [Hinze, Pinnau, Ulbrich, and Ulbrich, 2010].

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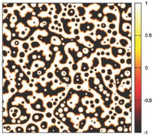

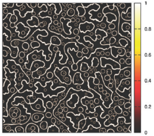

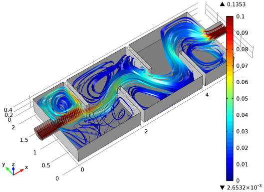

Navier–Stokes Control Problem

Given $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, $\nu > 0$, and $\beta > 0$, we consider the following stationary Navier–Stokes Control Problem

$$\min_{\vec{v}, \vec{u}} \mathcal{F}(\vec{v}, \vec{u}) = \min_{\vec{v}, \vec{u}} \frac{1}{2} \|\vec{v} - \vec{v}_d\|^2_{L^2(\Omega)} + \frac{\beta}{2} \|\vec{u}\|^2_{L^2(\Omega)}$$

subject to

$$\begin{cases} -\nu \nabla^2 \vec{v} + \vec{v} \cdot \nabla \vec{v} + \nabla p = \vec{u} + \vec{f} & \text{in } \Omega, \\ -\nabla \cdot \vec{v} = 0 & \text{in } \Omega, \\ \vec{v}(x) = \vec{g}(x) & \text{on } \partial\Omega. \end{cases}$$

Optimize-then-Discretize Strategy

KKT-conditions

In order to obtain the Optimality Conditions, we find the stationary points of the Lagrangian (using Fréchet derivative)

$$\mathcal{L}(\vec{v}, p, \vec{u}, \vec{\zeta}, \mu) = \mathcal{F}(\vec{v}, \vec{u}) + \int_\Omega \left(-\nu \nabla^2 \vec{v} + \vec{v} \cdot \nabla \vec{v} + \nabla p - \vec{u} - \vec{f} \right) \cdot \vec{\zeta} + \mu \nabla \cdot \vec{v} \, dx.$$

This leads to the gradient equation $\beta \vec{u} - \vec{\zeta} = 0$, and

$$\begin{cases} \left.\begin{array}{ll} -\nu \nabla^2 \vec{v} + \vec{v} \cdot \nabla \vec{v} + \nabla p = \frac{1}{\beta} \vec{\zeta} + \vec{f} & \text{in } \Omega, \\ -\nabla \cdot \vec{v} = 0 & \text{in } \Omega, \\ \vec{v}(x) = \vec{g}(x) & \text{on } \partial\Omega, \end{array}\right\} & \begin{array}{c}\text{state}\\ \text{equation}\end{array} \\ \left.\begin{array}{ll} -\nu \nabla^2 \vec{\zeta} - \vec{v} \cdot \nabla \vec{\zeta} + (\nabla \vec{v})^\top \vec{\zeta} + \nabla \mu = \vec{v}_d - \vec{v} & \text{in } \Omega, \\ -\nabla \cdot \vec{\zeta} = 0 & \text{in } \Omega, \\ \vec{\zeta}(x) = 0 & \text{on } \partial\Omega. \end{array}\right\} & \begin{array}{c}\text{adjoint}\\ \text{equation}\end{array} \end{cases}$$

Gauss–Newton Iteration

Given $\vec{v}^{(k)}, p^{(k)}, \vec{\zeta}^{(k)}, \mu^{(k)}$ approximation of $\vec{v}$, p, $\vec{\zeta}$, μ, we consider the following Gauss–Newton iterate:

$$\vec{v}^{(k+1)} = \vec{v}^{(k)} + \vec{\delta v}^{(k)}, \qquad p^{(k+1)} = p^{(k)} + \delta p^{(k)},$$
$$\vec{\zeta}^{(k+1)} = \vec{\zeta}^{(k)} + \vec{\delta\zeta}^{(k)}, \qquad \mu^{(k+1)} = \mu^{(k)} + \delta\mu^{(k)},$$

with

$$\begin{cases} \nu(\nabla\vec{\delta v}^{(k)}, \nabla\vec{w}) + (\vec{v}^{(k)} \cdot \nabla\vec{\delta v}^{(k)}, \vec{w}) + (\vec{\delta v}^{(k)} \cdot \nabla\vec{v}^{(k)}, \vec{w}) \\ \qquad -(\delta p^{(k)}, \nabla \cdot \vec{w}) - \frac{1}{\beta}(\vec{\delta\zeta}^{(k)}, \vec{w}) = \vec{R}_1^{(k)}, \\ \qquad -(q, \nabla \cdot \vec{\delta v}^{(k)}) = r_1^{(k)}, \\ (\vec{\delta v}^{(k)}, \vec{w}) + \nu(\nabla\vec{\delta\zeta}^{(k)}, \nabla\vec{w}) - (\vec{v}^{(k)} \cdot \nabla\vec{\delta\zeta}^{(k)}, \vec{w}) \\ \qquad +((\nabla\vec{v}^{(k)})^\top \vec{\delta\zeta}^{(k)}, \vec{w}) - (\delta\mu^{(k)}, \nabla \cdot \vec{w}) = \vec{R}_2^{(k)}, \\ \qquad -(q, \nabla \cdot \vec{\delta\zeta}^{(k)}) = r_2^{(k)}. \end{cases}$$

Numerical Discretization

In the following, we employ inf-sup stable finite element pairs, with:

- $\mathbf{M}$ (resp., M) is the mass matrix on the velocity (resp., pressure) space; we have that $M, \mathbf{M} \succ 0$;
- $\mathbf{K}$ (resp., K) is a FE discretization of $-\nabla^2$ on the velocity (resp., pressure) space; we have that $\mathbf{K} \succ 0$, $K \succeq 0$;
- $\mathbf{N}^{(k)}$ is a FE discretization of $\vec{v}^{\,(k)} \cdot \nabla$; we have that $\mathbf{N}^{(k)}$ is skew-symmetric if $\nabla \cdot \vec{v}^{\,(k)} = 0$;
- $\mathbf{H}^{(k)}$ is the matrix arising from second-order informations on the convection term $\vec{v}^{\,(k)} \cdot \nabla$;
- B is the (negative) divergence matrix.

Discretized System

Upon discretizing the previous system of PDEs, one obtains

$$\underbrace{\begin{bmatrix} \Phi^{(k)} & \Psi^\top \\ \Psi & -\Theta \end{bmatrix}}_{\mathcal{A}^{(k)}} \begin{bmatrix} \delta \boldsymbol{v}^{(k)} \\ \delta \boldsymbol{\zeta}^{(k)} \\ \delta \boldsymbol{\mu}^{(k)} \\ \delta \boldsymbol{p}^{(k)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_2^{(k)} \\ \boldsymbol{R}_1^{(k)} \\ \boldsymbol{r}_1^{(k)} \\ \boldsymbol{r}_2^{(k)} \end{bmatrix},$$

where

$$\Phi^{(k)} = \begin{bmatrix} \mathbf{M} & \mathbf{D}_{\mathrm{adj}}^{(k)} \\ \mathbf{D}^{(k)} & -\frac{1}{\beta}\mathbf{M} \end{bmatrix}, \quad \Psi = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \quad \Theta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Here, we set

$$\mathbf{D}^{(k)} = \nu \mathbf{K} + \mathbf{N}^{(k)} + \mathbf{H}^{(k)} \quad \mathbf{D}_{\mathrm{adj}}^{(k)} = \nu \mathbf{K} - \mathbf{N}^{(k)} + (\mathbf{H}^{(k)})^\top.$$

Preconditioning Approach

Preconditioning for Saddle Point Systems

Given the invertible system

$$\mathcal{A}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} \Phi & \Psi_1 \\ \Psi_2 & -\Theta \end{bmatrix},$$

if we precondition it with an invertible preconditioner of the form:

$$\mathcal{P} = \begin{bmatrix} \Phi & 0 \\ \Psi_2 & -S \end{bmatrix},$$

where $S = \Theta + \Psi_2 \Phi^{-1} \Phi_1$, the eigenvalues of the preconditioned matrix will be [Ipsen, 2001], [Murphy, Golub, and Wathen, 1999]

$$\lambda(\mathcal{P}^{-1}\mathcal{A}) = \{1\}.$$

In practice, we replace Φ and S with cheap approximation $\widetilde{\Phi}$ and $\widetilde{S}$.

Augmented Lagrangian Preconditioner

Given $\gamma > 0$ and suitable $\mathcal{W}$, we consider the following augmented Lagrangian preconditioner:

$$\mathcal{P}_\gamma = \left[\begin{array}{cc} \Phi^{(k)} + \gamma \Psi^\top \mathcal{W}^{-1} \Psi & \Psi^\top \\ 0 & -S_\gamma \end{array} \right],$$

where $S_\gamma = \Psi(\Phi + \gamma \Psi^\top \mathcal{W}^{-1} \Psi)^{-1} \Psi^\top$.
We employ as matrix $\mathcal{W}$ the following matrix:

$$\mathcal{W} = \left[\begin{array}{cc} 0 & W \\ W & 0 \end{array} \right],$$

with W being the pressure mass matrix M or its diagonal.
Rather than solving for the $(1,1)$-block and the Schur complement, we employ suitable cheap approximations of them.

Approximating $(1,1)$-block

As an approximate inverse of the $(1,1)$-block

$$\Phi^{(k)} + \gamma \Psi^\top \mathcal{W}^{-1} \Psi = \begin{bmatrix} \mathbf{M} & \mathbf{D}_{\mathrm{adj}}^{(k)} + \gamma B^\top W^{-1} B \\ \mathbf{D}^{(k)} + \gamma B^\top W^{-1} B & -\frac{1}{\beta}\mathbf{M} \end{bmatrix},$$

we employ a fixed number of GMRES iterations with preconditioner

$$\mathcal{P}_{(1,1)} = \begin{bmatrix} \mathbf{M} & 0 \\ \mathbf{D}^{(k)} + \gamma B^\top W^{-1} B & -\widetilde{S} \end{bmatrix}.$$

Here, $\widetilde{S}$ is the approximation of the inner Schur complement obtained with the "matching strategy" [Pearson and Wathen, 2012], given by

$$\widetilde{S} := (\mathbf{D}^{(k)} + \gamma B^\top W^{-1} B + \Lambda)\mathbf{M}^{-1}(\mathbf{D}_{\mathrm{adj}}^{(k)} + \gamma B^\top W^{-1} B + \Lambda),$$

where $\Lambda = \frac{1}{\sqrt{\beta}}\mathbf{M}$.

Approximating Schur Complement

By employing the Sherman–Morrison–Woodbury formula, one can write that

$$S_\gamma^{-1} = \left[\Psi(\Phi^{(k)})^{-1}\Psi^\top\right]^{-1} + \gamma\mathcal{W}^{-1}.$$

Approximating Schur Complement

By employing the Sherman–Morrison–Woodbury formula, one can write that

$$S_\gamma^{-1} = \left[\Psi(\Phi^{(k)})^{-1}\Psi^\top\right]^{-1} + \gamma \mathcal{W}^{-1}.$$

In order to have a robust preconditioner with respect to β, we consider the following approximation of S_γ^{-1}:

$$\widetilde{S}_\gamma^{-1} = \begin{bmatrix} K^{-1} & \gamma W^{-1} \\ \gamma W^{-1} & -\frac{1}{\beta} K^{-1} \end{bmatrix}.$$

Summarizing AL Preconditioner

- Krylov solver: FGMRES
  - Solve for $\mathcal{A}_\gamma^{(k)}$ with preconditioner $\widetilde{\mathcal{P}}_\gamma$
    - Applying $\widetilde{S}_\gamma^{-1}$
      - Approximate mass matrix inverse on the pressure space
      - Approximate stiffness matrix inverse on the pressure space
    - Krylov solver: FGMRES
      - Applying $\widetilde{\mathcal{P}}_{(1,1)}^{-1}$
        - Chebyshev semi-iteration for mass matrix on the velocity space
        - Applying $\widetilde{S}^{-1}$
          - 1 F-Cycle of AL multigrid

Numerical Results

Test 1 (Exact Solution)

We solve the stationary Navier–Stokes control problem in $\Omega = [-1, 1]^2$ with the following exact solution:

$$\vec{v} = \vec{v}_d = [xy^3, \frac{1}{4}(x^4 - y^4)]^\top, \qquad \vec{\zeta} = [0, 0]^\top,$$

for different values of ν and β. For a given β, we set $\gamma = 10\beta^{-0.5}$.

Test 1 (Exact Solution)

Table: Degrees of freedom (DoF) and average GMRES iterations of the augmented Lagrangian preconditioner with $\gamma = 10\beta^{-0.5}$, for $\nu = \frac{1}{100}$, $\frac{1}{500}$, and $\frac{1}{1000}$, and a range of l, β.

l	DoF	$\nu = \frac{1}{100}$			$\nu = \frac{1}{500}$			$\nu = \frac{1}{1000}$		
		β			β			β		
		10^{-3}	10^{-4}	10^{-5}	10^{-3}	10^{-4}	10^{-5}	10^{-3}	10^{-4}	10^{-5}
1	484	5	5	5	5	5	5	5	5	5
2	1796	4	4	4	3	4	4	3	4	4
3	6916	4	5	4	5	4	5	4	4	4
4	27,140	4	5	5	7	6	5	7	6	5
5	107,524	5	5	5	7	7	7	10	9	7
6	428,036	5	5	5	8	9	10	21	10	7

Test 1 (Exact Solution)

Table: Number of Gauss–Newton iterations required for stationary Navier–Stokes control problem. In each cell are the Gauss–Newton iterations for the given l, ν, and $\beta = 10^{-j}$, $j = 3, 4, 5$.

l	$\nu = \frac{1}{100}$			$\nu = \frac{1}{500}$			$\nu = \frac{1}{1000}$		
1	2	2	2	2	2	2	2	2	2
2	3	2	2	2	2	2	2	2	2
3	3	2	2	3	2	3	3	2	3
4	3	3	2	3	2	3	3	2	2
5	3	3	2	3	3	2	3	3	3
6	3	3	2	3	3	3	3	3	2

Test 2

We solve the stationary Navier–Stokes control problem in $\Omega = [-1, 1]^2$ with

$$\vec{f} = [0, 0]^\top, \qquad \vec{v}_d = [0, 0]^\top,$$

for different values of ν and β. For a given β, we set $\gamma = 10\beta^{-0.5}$.

Test 2

Table: Degrees of freedom (DoF) and average GMRES iterations of the augmented Lagrangian preconditioner with $\gamma = 10\beta^{-0.5}$, for $\nu = \frac{1}{100}$, $\frac{1}{500}$, and $\frac{1}{1000}$, and a range of l, β.

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l	DoF	10^{-3}	10^{-4}	10^{-5}	10^{-3}	10^{-4}	10^{-5}	10^{-3}	10^{-4}	10^{-5}
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2	1796	4	4	4	3	4	3	4	4	4
3	6916	5	6	5	5	5	4	5	5	4
4	27,140	5	6	6	7	6	5	7	6	6
5	107,524	7	7	5	7	9	8	10	8	8
6	428,036	9	6	7	13	7	12	14	6	10

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Table: Number of Gauss–Newton iterations required for stationary Navier–Stokes control problem. In each cell are the Gauss–Newton iterations for the given l, ν, and $\beta = 10^{-j}$, $j = 3, 4, 5$.

l	$\nu = \frac{1}{100}$			$\nu = \frac{1}{500}$			$\nu = \frac{1}{1000}$		
1	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
3	2	2	2	2	2	2	2	2	2
4	2	2	2	2	2	2	2	2	2
5	3	2	2	2	2	2	2	2	2
6	2	2	2	3	2	2	3	2	2

Conclusions

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- Can efficiently solve Navier–Stokes control problems with inexact Newton;
- preconditioner based on potent augmented Lagrangian approach;
- ongoing work & challenges:
  - solve more complex PDEs;
  - solve boundary control problems;
  - consider different cost functionals;
  - add algebraic constraints on state/control variables (IPMs).

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Thank you